128

ASYMPTOTICS OF SOME THREE AND FOUR SPECIES MODELS

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KUNWAR SINGH

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SCHOOL OF ENVIRONMENTAL SCIENCES
JAWAHARLAL NEHRU UNIVERSITY
NEW DELHI- 110067, INDIA

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JAWAHARLAL NEHRU UNIVERSITY
NEW DELHI – 110 067

School of Environmental Sciences

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CERTIFICATE

Certified that this dissertation entitled "**Asymptotics of some three and four species models**" submitted by **KUNWAR SINGH** in partial fulfilment of the requirements for the award of the degree of Master of Philosophy of Jawaharlal Nehru University is his own work and has not been previously submitted for any other degree of this or any other university.

KUNWAR SINGH

Prof. L. K PANDE
(Supervisor)

Prof. J. SUBBA RAO
(Dean)

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CONTENTS

CHAPTER- 1

INTRODUCTION

The study of the four species models occupies an important place in theoretical biology. The elucidation of these models will lead to clues to an understanding of the more complex multispecies systems. The ecosystem models, as described by a set of differential equations, are in general non linear. It is difficult to obtain exact analytical solutions in these cases. Any information obtainable using either any analytical or numerical methods or a combination of both is therefore very useful. It is easier to obtain such information for models with lower number of interacting species. Such information may then provide clues to results for more complex systems.

In this dissertation we have discussed four species ecosystem model within the frame work of Lotka Velterra model. We have analysed the one prey-three predator system in which the prey-predator interaction and self interaction for the prey are considered.

Next we have considered the three prey-one predator system in which prey-predator interaction and self interaction for the predator are included.

The details of these models and results claimed on them are discussed in chapter 3. In chapter 4 numerical solution using rung kutta method are obtained. Chapter 5 gives our main conclusions. The methods used by us were developed in certain three species systems with Lotka-Volterra interactions, which were stimulated by similar three species models with Gompertz interaction which lead to exact solution. These three species are reviewed in chapter 2.

CHAPTER- 2
(Section- A)

REVIEW OF SOME THREE SPECIES MODELS

We discuss the Gompertz models for some of the three species ecosystems. For instance we consider a food chain system. Let N_1, N_2 and N_3 be the three populations with N_2 preying on N_1 and N_3 preying on N_2. The time development of these populations will be governed by the various self interactions and mutual interaction terms. All these interaction terms are written in Gompertz form. The equations describing the model are:

$$\dot{N}_1 = \in_1 N_1 - \alpha_1 N_1 \log N_1 - \beta_1 N_1 \log N_2$$

$$\dot{N}_2 = -\in_2 N_2 - \beta_2 N_2 \log N_2 + \alpha_2 N_2 \log N_1 - r_2 N_2 \log N_3 \qquad \ldots(1)$$

$$\dot{N}_3 = -\in_3 N_3 - \alpha_3 N_3 \log N_3 + \beta_3 N_3 \log N_2$$

Where $\dot{N}_1$, $\dot{N}_2$ and $\dot{N}_3$ stand for the respective time derivatives. The signs of various terms depend on whether they represent self interaction, or prey predator interaction. The sign is negative for the first one and as for the latter terms, there is a negative sign in the equation for time development of prey population and a positive sign in the corresponding equation for the predator population. The $\in_1$ term is here the natural growth term and $\in_2$ and $\in_3$ are decay terms. Terms carrying the constants α_1, β_2 and α_3 are self interaction terms and remaining terms represent the prey predator interactions.

Introducing the notations

$$X_1 = \log N_1; X_2 = \log N_2, X_3 = \log N_3;$$

We can rewrite equations (1) as

$$\dot{X}_1 = \in_1 - \alpha_1 X_1 - \beta_1 X_2.$$

$$\dot{X}_2 = -\in_2 - \beta_2 X_2 + \alpha_2 X_1 - \gamma_2 X_3 \qquad \ldots(2)$$

$$\dot{X}_3 = -\in_3 - \alpha_3 X_3 + \beta_3 X_2.$$

The above was the general situation where we considered all the different types of interactions. Out interest is to see what happens when only pre-predator interactions and self interaction for population N_2 are there.

So we have

$$\alpha_1 = \alpha_3 = 0$$

Thus equation (2) reduces to

$$\dot{X}_1 = \in_1 - \beta_1 X_2 \qquad \ldots(3)$$

$$\dot{X}_2 = -\in_2 - \beta X_2 + \alpha_2 X_1 - \gamma_2 X_3 \qquad \ldots(4)$$

$$\dot{X}_3 = -\in_3 + \beta_3 X_2 \qquad \ldots(5)$$

First differentiating equation (4) and then substituting the values of $\dot{X}_1$ and $\dot{X}_3$ from equation (3) and (5),

we get

$$\ddot{X}_2 = A - B X_2 - \beta_2 \dot{X}_2 \qquad \ldots(6)$$

where $A = \alpha_2 \in_1 + \gamma_2 \in_3$

$$B = \alpha_2 \beta_1 + \gamma_2 \beta_3$$

equation (6) is non homogenous linear equation, the full solution of which is

$$X_2 = \frac{A}{B} + D_1 e^{E_1 t} + D_2 e^{E_2 t} \qquad \ldots(7)$$

Where D_1 and D_2 are two arbitrary constants and

$$E_1 = \frac{-B_2 + \sqrt{B_2^{\,2} - 4B}}{2} \qquad \text{...(8)}$$

$$E_2 = \frac{-B_2 - \sqrt{B_2^{\,2} - 4B}}{2}$$

Substituting the value of X_2 from equation (7) in equation (3) and (5) we get

$$X_1 = C_1 + K_1\, t - \beta_1\left(\frac{D_1}{E_1}e^{E_1 t} + \frac{D_2}{E_2}e^{E_2 t}\right) \qquad \text{....(9)}$$

$$X_3 = C_2 + \frac{\alpha_2}{\gamma_2}\, Kt + \beta_3\left(\frac{D_1}{E_1}e^{E_1 t} + \frac{D_2}{E_2}e^{E_2 t}\right) \qquad \text{...(10)}$$

Where $K = \frac{r_2\,(\beta_3 \in_1 - \beta_1 \in_3)}{B}$ and

C_1 and C_2 are integration constants. It is clear from equation (8) that E_1 and E_2 always have negative parts. Therefore, X_2 (and hence N_2) is always finite and non vanishing.

For $t \to \infty$, it acquires the value

$$X_2\,(t \to \infty) = \frac{A}{B}$$

The value of X_1 and X_3 are governed by the condition on $(\beta_3 \in_1 = \beta_1 \in_3)$.

$(\beta_3 \in_1 - \beta_1 \in_3) > 0$

$N_1\,(t \to \infty) \to \infty$

$N_3(t \to \infty) \to \infty$

Condition $(\beta_3 \in_1 - \beta_1 \in_3) < 0$

$N_1\,(t \to \infty) \to 0$

$N_3\,(t \to \infty) \to 0$

Condition $(\beta_3 \in_1 - \beta_1 \in_3) = 0$

$N_1(t \to \infty) \to C_1$

$N_3\,(t \to \infty) \to C_2$

Hence N_1 and N_3 remain finite and coexist.

We see the following results.

i. Under condition (i), population N_1 and N_3 rise indefinitely and population N_2 oscillates and then settles to finite value.

ii. Under condition (ii), populations N_1 and N_3 vanishes and population N_2 settles to finite value.

iii In this case, there is coexistence of all the three populations.

It is found that under suitable choice of parameter $(\beta_3 \in_1 - \beta_1 \in_3 > 0)$ both N_1 and N_3 rose indefinitely while N_2 reached a finite constant value asymptotically. Here the two rising populations behave as if they are growing in an unlimited environment without any apparent interaction between them. Yet in reality they are linked via intermediate quiescent population.

Under condition (ii), population N_1 and N_3 vanishes, which is clearly a rather unphysical situation, since this would mean that despite vanishing of the prey population N_1, the predator population N_2 continues to be finite. This results shown up because our interaction are not fully causal. This results can be improved by incorporating appropriate time-lags in all our interactions.

CHAPTER- 2
(Section- B)

REVIEW OF ASSYMPTOTICS OF SOME THREE-SPECIES MODELS

Here the asymptotic behavior of component populations in a few interacting three species models is derived. This is done by exploiting a constraint that exists in the subspace of two of the three population, and by using Lauent serious expansions in the asymptotic region in a suitable variable.

The Three-Species Food Chain Model

The model consisted of three populations N_1, N_2 and N_3 with N_2 praying on N_1 and N_3 preying on N_2. Besides containing the appropriate prey-predator interactions, the model also contained self-interaction for population N_2.

Equations

$$\dot{N}_1 = \in_1 N_1 - \beta_1 N_1 N_2 \qquad \ldots\ldots(1)$$

$$\dot{N}_2 = -\in_2 N_2 - \beta_2 N_2^2 + \alpha_2 N_1 N_2 - r_2 N_2 N_3 \qquad \ldots\ldots(2)$$

$$\dot{N}_3 = -\in_3 N_3 + \beta_3 N_2 N_3 \qquad \ldots\ldots(3)$$

Where all the parameters $\in_1$, β_1, etc, are positive.

In terms of the variable Z defined by

$$Z = e^{\delta t}, \delta > 0 \qquad \ldots(4)$$

These equations become

$$\delta Z \frac{dN_1}{dZ} = \in_1 N_1 - \beta_1 N_1 N_2 \qquad \ldots(1a)$$

$$\delta Z \frac{dN_2}{dZ} = -\epsilon_2 N_2 - \beta_2 N_2^2 + \alpha_2 N_1 N_2 - \gamma_2 N_2 N_3 \qquad \text{...(2a)}$$

$$\delta Z \frac{dN_3}{dZ} = -\epsilon_3 N_3 + \beta_3 N_2 N_3 \qquad \text{...(3a)}$$

On equating the expression for N_2 from (1a) and (3a), we get

$$-\frac{1}{\beta_1}\left\{ \delta Z \frac{d}{dZ}(\log N_1) - \epsilon_1 \right\} = \frac{1}{\beta_3}\left\{ \delta Z \frac{d}{dZ}(\log N_3) + \epsilon_3 \right\} \qquad \text{...(5)}$$

which on integration leads to,

$$N_1^{B_3} N_3^{B_1} = KZ^{\sigma/\delta} \qquad \text{...(6)}$$

Where K is constant and $\sigma = \epsilon_1\beta_3 - \epsilon_3\beta_1$.

The pressure of self interaction term in equation (2) generally leads to frictional damping and saturation. Therefore we look for a solution of the system of equations such that

$N_2 \to$ constant as $t \to \infty$

Or

$$\lim_{Z\to\infty} N_2(Z) = b_0 \text{ where } b_0 \text{ is a constant} \qquad \text{...(7)}$$

So, around $Z = \infty$, the Laurent expansion

$$N_2(Z) = b_0 + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \qquad \text{...(8)}$$

We then have

$$\lim_{Z\to\infty} Z \frac{d}{dZ}(\log N_2) = 0$$

Equation (2a) can be written as

$$\delta Z \lim_{Z\to\infty} Z \frac{d}{dZ} (\log N_2) = -\epsilon_2 - \beta_2 N_2 + \alpha_2 N_1 - r_1 N_3 \qquad \text{.....(10)}$$

we get

$$\lim_{Z\to\infty} \{\alpha_2 N_1(Z) - \gamma_2 N_3(Z)\} = \in_2 + \beta_2 b_0 \equiv C \qquad ...(11)$$

Where C must be constant. Thus the laurent expansions of $N_1(Z)$ and $N_3(Z)$ around $Z = \infty$ should be

$$N_1(Z) = \gamma_2 f(Z) + a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \qquad ...(12)$$

$$N_3(Z) = \alpha_2 f(Z) + c_0 + \sum_{n=1}^{\infty} c_{-n} Z^{-n}. \qquad ...(13)$$

Where f(Z) is as yet an unspecified function of Z which does not vanish as $Z \to \infty$.

Substituting these now in (5) for $Z \to \infty$, we get

$$[\gamma_2 f(Z) + a_0 + ...]^{\beta_3} [\alpha_2 f(Z) + C_0 + ...]^{\beta_1} = KZ^{\sigma/\delta} \qquad ...(14)$$

Now three cases arise.

CASE 1: $\sigma > 0$

An examination of (14) reveals that the leading behavior of $f(Z)$ as $Z \to \infty$ must be Z^m for some $m > 0$. Thus

$$\delta m = \frac{\sigma}{\beta_1 + \beta_2}$$

We may new choose to replace δ by $\delta' = \delta m$, and work with variable $Z' = e^{\delta' t}$ instead. The leading behavior of $f(Z')$is then simply given by the term KZ' where K is a constant. For convenience we drop the prime on δ'. Thus assymptotic expansion for $N_1(Z)$ and $N_3(Z)$ thus take the form.

$$N_1(Z) = a_1 Z + a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}. \qquad ...(16)$$

$$N_3(Z) = C_1 Z + C_0 + \sum_{n=1}^{\infty} C_{-n} Z^{-n}. \qquad \ldots(17)$$

Substituting equation (8), (16), (17) in (1 a)- (3 a) and equating the coefficients of like powers of Z we obtain.

$$c_1 = a_1(\alpha_2/\gamma_2), \quad b_0 = (\in 1 + \in_3)/(\beta_1 + \beta_3)$$

$$a_0 = \frac{(\in_2 + \beta_2 b_0)\beta_1}{(\beta_1 + \beta_3)}, \quad c_0 = \frac{-\alpha_2 \beta_3}{\gamma_2 \beta_1} a_0.$$

Reverting to the variable t:

$$\lim_{t\to\infty} N_1(t) = a_1 e^{\sigma' t}$$

$$\lim_{t\to\infty} N_2(t) = (\in_1 + \in_3)/(\beta_1 + \beta_3)$$

$$\lim_{t\to\infty} N_3(t) = \left(\frac{\alpha_2}{\gamma_2}\right) a_1 e^{\sigma' t}$$

where $\sigma' = \sigma/(\beta_1 + \beta_3) > 0$

CASE 2 : $(\sigma < 0)$

Since the r.h.s of (14) must vanish in the limit $Z \to \infty$, the function $f(Z)$ must be identically zero and furthermore a_0 and c_0 can not both be nonzero, only one of them can. The appropriate expansions in this case

$$N_1 = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}.$$

$$N_2 = b_0 + \sum_{n=1}^{\infty} b_{-n} Z^{-n}$$

$$N_3 = \sum_{n=1}^{\infty} C_{-n} Z^{-n}.$$

On substituting (22) in (1 a) – (3 a) we can get the value of a_0 and b_0.

Hence the selection

$$\lim_{t\to\infty} N_1(t) = a_0 = \frac{(\epsilon_1 \beta_2 + \epsilon_2 \beta_1)}{\alpha_2 \beta_1}$$

$$\lim_{t\to\infty} N_2(t) = b_0 = \epsilon_1 / \beta_1$$

$$\lim_{t\to\infty} N_3(t) = b_0 = C_{-1}\, e^{(\sigma/\beta_1)t} \to 0$$

CASE-3: ($\sigma = 0$)

In this case, equations (5) reduces to

$$N_1^{B_3} N_3^{B_1} = K$$

Which in view of equation (11) implies the behaviour

$$N_1(Z) = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}$$

$$N_3(Z) = c_0 + \sum_{n=1}^{\infty} c_n Z^{-n}.$$

Substituting these expressions in (1 a) – (3 a) and equating coefficients of like terms,

We get

$$b_0 = \frac{\epsilon_1}{\beta_1} = \frac{\epsilon_3}{\beta_3},$$

while a_0 and c_0 are given by the solutions of the equations

$$\alpha_2 a_0 = \gamma_2 c_0 = (\epsilon_1 \beta_2 + \epsilon_2 \beta_1)/\beta_1.$$

$$a_0^{\beta_3} c_0^{\beta_1} = K$$

Solution

$$\lim_{t\to\infty} N_1 = a_0 .$$

$$\lim_{t\to\infty} N_2 = b_0 = \frac{\epsilon_1}{\beta_1}$$

$$\lim_{t\to\infty} N_3 = C_0.$$

2. THE ONE PREY- TWO PREDATOR MODEL

The model consisted of three populations N_1, N_2 and N_3 with N_2 and N_3 preying on N_1. Besides containing the appropriate prey-predator interactions, the model also contained self-interaction for the population N_1.

Equations

$$\dot{N}_1 = \epsilon_1 N_1 - \alpha_1 N_1 N_2 - \beta_1 N_1 N_2 - \gamma_1 N_1 N_3.$$

$$\dot{N}_2 = -\epsilon_2 N_2 + \alpha_2 N_1 N_2.$$

$$\dot{N}_3 = -\epsilon_3 N_3 + \alpha_3 N_1 N_3.$$

In terms of the variable Z, we new have the constraint

$$N_2^{\alpha_3} N_3^{-\alpha_2} = LZ^{\eta/\delta}$$

Where L is constant determined by the initial conditions and

$$\eta = \alpha_2 \epsilon_3 - \alpha_3 \epsilon_2.$$

Now two possibilities axises

CASE-1 : $(\eta > 0)$

Here solution is

$$\lim_{t\to\infty} N_1(t) = \frac{\epsilon_2}{\alpha_2}$$

$$\lim_{t\to\infty} N_2(t) = \frac{\alpha_2 \in_1 - \alpha_1 \in_2}{\alpha_2 \beta_1}$$

$$\lim_{t\to\infty} N_3(t) = 0.$$

CASE-2: ($\eta < 0$)

Solution: is

$$\lim_{t\to\infty} N_1(t) = \frac{\in_3}{\alpha_3}$$

$$\lim_{t\to\infty} N_2(t) = 0$$

$$\lim_{t\to\infty} N_3(t) = (\alpha_3 \in_1 - \alpha_1 \in_3)/\alpha_3\gamma_1.$$

CASE-3 : ($\eta = 0$)

Solution

$$\lim_{t\to\infty} N_1(t) = \frac{\in_2}{\alpha_2}$$

$$\lim_{t\to\infty} N_2(t) = b_0$$

$$\lim_{t\to\infty} N_3(t) = c_0$$

Where

$$\beta_1 b_0 + \gamma_1 c_0 = (\alpha_2 \in_1 - \alpha_1 \in_2)/\alpha_2$$

3. THE TWO PREY-ONE PREDATOR MODEL

The model consisted of three populations N_1, N_2 and N_3 with N_3 preying on N_1 and N_2. Besides containing the appropriate prey-predator interactions, the model also contained self-interaction for populations N_3.

Equation

$$\dot{N}_1 = \epsilon_1 N_1 - \gamma_1 N_1 N_3$$

$$\dot{N}_2 = \epsilon_2 N_2 - \gamma_2 N_2 N_3$$

$$\dot{N}_3 = -\epsilon_3 N_3 - \gamma_3 N_3^2 + \alpha_3 N_1 N_3 + \beta_3 N_2 N_3.$$

The constraint in terms of the variable $\cancel{Z}$, is now $N_1^{\gamma_2} n_2^{-\gamma_1} = MZ^{w/\delta}$.

Where M is initial-condition dependant constant and

$$w = \epsilon_1 \gamma_2 - \epsilon_2 \gamma_1.$$

CASE-1 : (W > 0)

The assymptotics are now given as follows

$$\lim_{t\to\infty} N_1(t) = \frac{(\epsilon_3 \gamma_1 + \epsilon_1 \gamma_3)}{\alpha_3 \gamma_1}$$

$$\lim_{t\to\infty} N_2(t) = 0$$

$$\lim_{t\to\infty} N_3(t) = \frac{\epsilon_1}{\gamma_1}$$

CASE – 2: (w< 0)

In this case we get

$$\lim_{t\to\infty} N_1(t) = 0$$

$$\lim_{t\to\infty} N_2(t) = (\epsilon_3\,\gamma_2 + \epsilon_2\,\gamma_3)/\beta_3\gamma_2.$$

$$\lim_{t\to\infty} N_3(t) = \frac{\epsilon_2}{\gamma_2}$$

CASE – 3: (w = 0)

In this case, we get

$$\lim_{t\to\infty} N_1(t) = a_0$$

$$\lim_{t\to\infty} N_2(t) = b_0$$

$$\lim_{t\to\infty} N_3(t) = \frac{\epsilon_1}{\gamma_1},$$

where

$$\alpha_3\,a_0 + \beta_3\,b_0 = (\epsilon_3\,\gamma_1 + \epsilon_1\,\gamma_3)/\gamma_1.$$

The methods used here as a basis for discussion of certain four species models in the next chapter.

CHAPTER - 3

ANALYSIS OF ASSYMPTOPTICS OF SOME FOUR SPECIES MODELS

In this chapter we carry out an analysis of assymptotics of certain four species ecosystems. In section I we consider the three prey-one predator system in which prey-predator interaction and self interaction for the predator is considered. In section II we deal with the one prey-three predator interaction in which prey-predator interaction and self interaction for the prey is considered.

It is not possible to write the exact solutions of the above systems. However important information about the populations can be ascertained by analysing the behaviour of the systems in the assymptotic region as $t \to \infty$. The results are obtained by exploring the constraint that exist in the subspace of three populations and using suitable laurent series expansions in an appropriately choosen variable in the assymptotic region.

SECTION - A

THREE PREY-ONE PREDATOR SYSTEM

The model consisted of four populations N_1, N_2, N_3, N_4 with N_4 preying on N_1, N_2, N_3, Besides the model containing the prey-predator interaction, the model also contained self interaction for the population N_4.

Equation

$$\dot{N}_1 = a_1 N_1 - b_1 N_1 N_4 \qquad \ldots(1)$$

$$\dot{N}_2 = a_2 N_2 - b_2 N_2 N_4 \qquad \ldots(2)$$

$$\dot{N}_3 = a_3 N_3 - b_3 N_3 N_4 \qquad \ldots(3)$$

$$\dot{N}_4 = -a_4 N_4 - b_4 N_4^2 + c_4 N_1 N_4 + d_4 N_2 N_4 + e_4 N_3 N_4. \qquad \ldots(4)$$

where all the parameters a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 etc. are positive.

In terms of variable Z defined by

$$Z = e^{\delta t}$$

These equation becomes:

$$\delta Z \frac{dN_1}{dZ} = a_1 N_1 - b_1 N_1 N_4 \qquad \ldots(5)$$

$$\delta Z \frac{dN_2}{dZ} = a_2 N_2 - b_2 N_2 N_4 \qquad \ldots(6)$$

$$\delta Z \frac{dN_3}{dZ} = a_3 N_3 - b_3 N_3 N_4 \qquad \ldots(7)$$

$$\delta Z \frac{dN_4}{dZ} = - a_4 N_4 - b_4 N_4^2 + c_4 N_1 N_4 + d_4 N_2 N_4 + e_4 N_3 N_4 \qquad \ldots(8)$$

The presence of the self interaction term in equation (4) generally leads to frictional damping and saturation. Therefore we look for a solution of the system such that

$N_4 \to$ constant as $t \to \infty$

or $\lim_{Z \to \infty} N_4 = d_0$ ...(9)

where d_0 is constant. This would imply, around $Z = \infty$, the following Laurent expansion.

$$N_4 = d_0 + \sum_{n=1}^{\infty} d_{-n} Z^{-n}$$

Equation (8) can be written as

$$\delta Z \frac{d}{dZ} \{\log N_4(Z)\} = -a_4 - b_4 N_4 + c_4 N_1 + d_4 N_2 + e_4 N_3$$

or $$\lim_{Z \to \infty} \delta Z \frac{d}{dZ} \{\log N_4(X)\} = -a_4 - b_4 \lim_{Z \to \infty} N_4(Z) + c_4 \lim_{Z \to \infty} N_1(Z)$$

$$+ d_4 \lim_{Z \to \infty} N_2(Z) + e_4 \lim_{Z \to \infty} N_3(Z) \text{ ss}$$

or $$0 = -a_4 - b_4 d_0 + c_4 \lim_{Z \to \infty} N_1(Z) + d_4 \lim_{Z \to \infty} N_2(Z) + e_4 \lim_{Z \to \infty} N_3(Z)$$

or $$c_4 \lim_{Z \to \infty} N_1(Z) + d_4 \lim_{Z \to \infty} N_2(Z) + e_4 \lim_{Z \to \infty} N_3(Z)$$

$$= a_4 + b_4 d_0 = D$$

where D is positive

Since c_4, d_4, e_4 are positive and linear combination of N_1, N_2 and N_3 is also positive, hence f(Z) term ($\Sigma \beta_n Z^n$) in the Launent expansion will be absent.

Thus the laurent expansion of N_1, N_2 and N_3 around $Z = \infty$ should be;

$$N_1 = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \qquad \ldots(10)$$

$$N_2 = b_0 + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \qquad \ldots(11)$$

$$N_3 = c_0 + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \qquad \ldots(12)$$

On equating the expression for N_4 from equation (5) and (6), we get

$$\frac{1}{b_1}\{\delta Z \frac{d}{dZ}(\log N_1) - a_1\} = \frac{1}{b_2}\{\delta Z \frac{d}{dZ}(\log N_2) - a_2\}$$

which on integration leads to the equation

$$N_1^{b_2} N_2^{-b_1} = k Z^{(a_1 b_2 - b_1 a_2)/\delta} \qquad \ldots(13)$$

$$N_1^{b_2} N_2^{-b_1} = k Z^{\sigma/\delta}$$

On equating the expression for N_4 from equation (5) and (7), we get

$$\frac{1}{b_1}\{\delta Z \frac{d}{dZ}(\log N_1) - a_1\} = \frac{1}{b_3}\{\delta Z \frac{d}{dZ}(\log N_3) - a_3\}$$

which on integration leads to equation

$$N_1^{b_3} N_3^{-b_1} = L Z^{\eta/\delta} \qquad \ldots(14)$$

where $\eta = a_1 b_3 - b_1 a_3$

CASE - I

$(\sigma > 0,\ \eta > 0)$

Rewriting equation (13) and (14)

$$N_1^{b_2} N_2^{-b_1} = kZ^{\sigma/\delta}$$

$$N_1^{b_3} N_3^{-b_1} = L Z^{\eta/\delta}$$

For $\sigma > 0$, $\eta > 0$ right hand side of above equations tends to infinite as $z \to \infty$. Hence constant term b_0 and c_0 in N_2 and N_3 must be zero.

Hence

$$N_1 = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \qquad ...(15)$$

$$N_2 = \sum_{n=2}^{\infty} b_{-n} Z^{-n} \qquad ...(16)$$

$$N_3 = \sum_{n=3}^{\infty} C_{-n} Z^{-n} \qquad ...(17)$$

Rewriting equation (5)

$$\delta Z \frac{dN_1}{d Z} = a_1 N_1 - b_1 N_1 N_4$$

$$\delta Z \frac{d}{dZ} (a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}) = a_1 (a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}) - b_1 (a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n})$$

$$(d_0 + \sum_{n=1}^{\infty} d_{-n} Z^{-n})$$

Comparing the constant term in above equation,

$a_1 a_0 - b_1 a_0 d_0 = 0$

$a_0 (a_1 - b_1 d_0) = 0$

Either $a_0 = 0$ or $(a_1 - b_1 d_0) = 0$

Since $a_0 \neq 0$,

hence $a_1 - b_1 d_0 = 0$

$$d_0 = \frac{a_1}{b_1}$$

Rewriting the equation (8)

$$\delta Z \frac{dN_4}{dZ} = -a_4 N_4 - b_4 N_4^2 + c_4 N_1 N_4 + d_4 N_1 N_4 + e_4 N_1 N_4$$

or $$\delta Z \frac{dN_4}{dZ} = -a_4 \, (d_0 + \sum_{n=1}^{\infty} d_{-n} \, Z^{-n}) - b_4(d_0 + \sum_{n=1}^{\infty} d_{-n}Z^{-n})^2 + c_4(a_0 + \sum_{n=1}^{\infty} a_{-n} \, Z^{-n})$$

$$(d_0 + \sum_{n=1}^{\infty} a_{-n} \, Z^{-n}) + d_4(\sum_{n=1}^{\infty} b_{-n}Z^{-n}) \, (d_0 + \sum_{n=1}^{\infty} c_{-n}Z^{-n}) + e_4(\sum_{n=1}^{\infty} c_{-n} \, Z^{-n}) \, (d_0 + \sum_{n=1}^{\infty} c_{-n} \, Z^{-n})$$

Comparing the constant term

$$- \, a_4 d_0 + -b_4 d_0^2 + c_4 a_0 d_0 = 0$$

or $$d_0 \, (a_0 \, c_4 - a_4 - b_4 \, d_0)$$

Since $d_0 \neq 0$

$$a_0 \, c_4 = a_4 + b_4 \, d_0$$

$$a_0 = \frac{a_4 \, b_1 + b_4 \, a_1}{b_1 \, c_4}$$

Hence Solution

$$\lim_{t \to \infty} N_1(t) = \frac{a_4 \, b_1 + a_1 \, b_4}{c_4 \, b_1}$$

$$\lim_{t \to \infty} N_2(t) = 0$$

$$\lim_{t \to \infty} N_3(t) = 0$$

$$\lim_{t \to \infty} N_4(t) = \frac{a_1}{b_1}$$

CASE - II

$(\sigma < 0, \eta < 0)$

Equations

$$N_1^{b_2} \, N_2^{-b_1} = K \, Z^{\sigma/\delta}$$

TH- 8086

$N_1^{b_3}\, N_3^{-b_1} + L\, Z^{\eta/\delta}$

For $\sigma < 0$, $\eta < 0$, right hand side of above equation tends to zero as $Z \to \infty$. So the constant term a_0 in N_1 must be zero.

Hence,

$$N_1 = \sum_{n=1}^{\infty} a_{-n}\, Z^{-n} \qquad \ldots(18)$$

$$N_2 = b_0 + \sum_{n=1}^{\infty} b_{-n}\, Z^{-n} \qquad \ldots(19)$$

$$N_3 = c_0 + \sum_{n=1}^{\infty} c_{-n}\, Z^{-n} \qquad \ldots(20)$$

Now substituting equation (18), (19), (20) in equation (5), (6), (7), (8) and equating coefficient of like powers of z.

We get,

$$d_0 = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$a_4 + b_4 d_0 = d_4 b_0 + e_4\, c_0$$

Hence the solution

$$\lim_{t\to\infty} N_1 = 0$$

$$\lim_{t\to\infty} N_2 = b_0$$

$$\lim_{t\to\infty} N_3 = c_0$$

$$\lim_{t\to\infty} N_4 = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

where $a_4 + b_4\, d_0 = d_4\, b_0 + e_4\, c_0$

CASE - III

$(\sigma > 0, \eta < 0)$

Re writing the equation (13) and (14)

$$N_1^{b_2} N_3^{-b_1} = K Z^{\sigma/\delta} \qquad \ldots(9a)$$

$$N_1^{b_3} N_3^{-b_1} = L Z^{\eta/\delta} \qquad \ldots(10a)$$

where $\sigma = a_1 b_2 - b_1 a_2$

$\eta = a_1 b_3 - b_1 a_3$

For $\sigma > 0$, right hand side of equation (9a) tends to ∞ as Z tends to ∞. Hence constant term b_0 in N_2 must be zero.

For $\eta < 0$, Right hand side of equation (10a) tends to zero as Z tends to infinite. Hence constant term a_0 in N_1 must be zero.

Let $N_1 = \frac{a_{-1}}{Z}$

$N_2 = \frac{b_{-1}}{Z}$

Substituting these in equation (9a)

$$a_{-1}^{b_2} b_{-1}^{-b_1} Z^{b_1 - b_2} = K Z^{\sigma/\delta}$$

Since $\sigma > 0$, hence b_1-$b_2 > 0$

New Conditions are

$\sigma > 0, \eta < 0,$ and $(b_1 - b_2) > 0$

$$N_1 = \frac{a_{-1}}{Z} \qquad \ldots(21)$$

$$N_2 = \frac{b_{-1}}{Z} \qquad \ldots(22)$$

$$N_3 = c_0 + \sum_{n=1}^{\infty} c_{-n}\, Z^{-n} \qquad \ldots(23)$$

$$N_4 = d_0 + \sum_{n=1}^{\infty} d_{-n}\, Z^{-n} \qquad \ldots(24)$$

Substituting the equations (19), (21), (22), (23), and (24) in equations (5) (6) (7) (8), comparing the coefficients of like powers of Z.

$$d_0 = \frac{a_3}{b_3}$$

$$e_4\, c_0 = a_4 + b_4\, d_0$$

$$c_0 = \frac{a_4\, b_3 + a_3\, b_4}{e_4\, b_3}$$

Hence Solution

$$\lim_{t \to \infty} N_1 = 0$$

$$\lim_{t \to \infty} N_2 = 0$$

$$\lim_{\to \infty} N_3 = \frac{a_4\, b_3 + a_3\, b_4}{e_4\, b_3}$$

$$\lim_{t \to \infty} N_4 = \frac{a_3}{b_3}$$

CASE - IV

($\sigma < 0$, $\eta > 0$)

Rewriting the equation (13) and (14)

$$N_1^{b_2} N_2^{-b_1} = K\, Z^{\sigma/\delta} \qquad \ldots.(9a)$$

$$N_1^{b_3} N_3^{-b_1} = L\, Z^{\eta/\delta} \qquad \text{....(10a)}$$

For $\sigma < 0$, Right hand side of equation (9a) tends to zero as $Z \to \infty$. Hence constant term a_0 in N_1 will be zero.

For $\eta > 0$ right hand side of equation (10a) tends to $Z \to \infty$. Hence constant term c_0 in N_3 will be zero.

Suppose $\quad N_1 = \dfrac{a_{-1}}{z}$

$$N_3 = \frac{c_{-1}}{z}$$

Substituting these in equation (10a)

$$a_{-1}^{b_3}\, c_{-1}^{-b_1}\; Z^{b_1 - b_3} = LZ^{\eta/\delta}$$

Since $\eta > 0$, so $b_1 - b_3 > 0$

New Conditions are $\quad \sigma < 0,\ \eta > 0,\quad$ and $\ b_1 - b_3 > 0$

Now

$$N_1 = \frac{a_{-1}}{z} \qquad \text{...(25)}$$

$$N_2 = b_0 + \sum_{n=1}^{\infty} b_{-1} z^{-n} \qquad \text{...(26)}$$

$$N_3 = \frac{c_{-1}}{z} \qquad \text{...(27)}$$

$$N_4 = d_0 + \sum_{n=1}^{\infty} d_{-n}\, Z^{-n} \qquad \text{...(28)}$$

Substituting the equations (25), (26), (27) and (28) in equations (5), (6), (7) and (8) and comparing the coefficients of powers of z.

$$d_0 = \frac{a_2}{b_2}$$

$d_4\ b_0 = a_4 + b_4 d_0$

Solution

$$\lim_{t\to\infty} N_1 = 0$$

$$\lim_{t\to\infty} N_2 = \frac{a_4 b_2 + a_2 b_4}{b_2\ d_4}$$

$$\lim_{t\to\infty} N_3 = 0$$

$$\lim_{t\to\infty} N_4 = \frac{a_2}{b_2}$$

SECTION - B

THE ONE PREY- THREE PREDATOR SYSTEM

The model consisted of four populations N_1, N_2, N_3 and N_4 with N_2, N_3, N_4 preying on N_1. Besides the model containing the prey- predator interaction the model also contained self interaction for the population N_2.

Equations

$$N_1 = a_1, N_1 - b_{1,} N_1^2 - C_1 N_1 N_2 - d_1 N_1 N_3 - e_1 N_1 N_4 \quad \ldots(1)$$

$$N_2 = -a_2 N_2 + b_2 N_2 N_1 \quad \ldots(2)$$

$$N_3 = -a_3 N_3 + b_3 N_1 N_3 \quad \ldots(3)$$

$$N_4 = -a_4 N_4 + b_4 N_1 N_4 \quad \ldots(4)$$

where all the parameters a_1, b_1, c_1, d_1 etc are positive.

In terms of variable Z defined by

$$Z = e^{\delta t} \quad , \delta > 0$$

these Equations become

1. $$S Z \frac{dN_1}{dZ} = a_1 N_1 - b_1 N_1^2 - c_1 N_1 N_2 - d_1 N_1 N_3 - e_1 N_1 N_4$$

2. $$S Z \frac{dN_2}{dZ} = - a_2 N_2 + b_2 N_1 N_2$$

3. $$S Z \frac{dN_3}{dZ} = -a_3 N_3 + b_3 N_1 N_3$$

4. $$S Z \frac{dN_4}{dZ} = - a_4 N_4 + b_4 N_1 N_4.$$

The presence of self interaction term in equation (I) generally leads to frictional damping and saturation. Therefore we look for a solution of the system of equation such that

$N_1 \to$ constant as $t \to \infty$.

or $\lim\limits_{Z \to \infty} N_1(Z) = a_0$ (constant)

Then Around $Z = \infty$, the laurent expansion.

$$N_1 = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}$$

We then have

$$\lim_{Z \to \infty} Z \frac{d}{dZ} \{\log N_1 (Z) = 0$$

From equation (5)

$$\delta Z \frac{dN_1(Z)}{dZ} = a_1 N_1 - b_1 N_1 N_1 - c_1 N_1 N_2 - d_1 N_1 N_3 - e_1 N_1 N_4.$$

or
$$\delta Z \frac{d\{\log N_1(Z)\}}{dZ} = a_1 - b_1 N_1 - c_1 N_2 - d_1 N_3 - e_1 N_4.$$

or
$$\lim_{Z \to \infty} \delta Z \frac{d\{\log N_1(Z)\}}{dZ} = a_1 - b_1 \lim_{Z \to \infty} N_1$$

$$- \lim_{Z \to \infty} [c_1 \lim_{Z \to \infty} N_2 + d_1 \lim_{Z \to \infty} N_3 + e_1 \lim_{Z \to \infty} N_4].$$

or
$$0 = a_1 - b_1 a_0 - \lim_{Z \to \infty} [c_1 \lim_{Z \to \infty} N_2 + d_1 \lim_{Z \to \infty} N_3 + e_1 \lim_{Z \to \infty} N_4]$$

$$[c_1 \lim_{Z \to \infty} N_2 + d_1 \lim_{Z \to \infty} N_3 + e_1, \lim_{Z \to \infty} N_4]$$

$$= a_1 - a_0 b_1 = C$$

Where C must be greater then zero. Since c_1, d_1, e_1 are positive and positive linear combination of N_1, N_2 and N_3 is also positive.

hence f(z) term ($\left(\Sigma \in_n Z^n\right)$ in the laurent expansion will be absent.

Thus the laurent expansion of N_2, N_3 and N_4 around $Z = \infty$ should be

$$8(b)\quad \begin{cases} N_2 = b_0 + \sum\limits_{Z=1}^{\infty} b_n Z^{-n} \\ N_3 = c_0 + \sum\limits_{Z=1}^{\infty} C_{-n} Z^{-n} \\ N_4 = d_0 + \sum\limits_{Z=1}^{\infty} d_{-n} Z^{-n} \end{cases}$$

On equating the expression for N_1 from equation (6) and (7), we get

$$\frac{1}{b_2}\left[\delta Z \frac{d}{dZ}\{\log N_2\} + a_2\right] = \frac{1}{b_3}\left[\delta Z \frac{d}{dZ}(\log N_3) + a_3\right]$$

Solving the above equation

$$N_2^{b_2} N_3^{-b_3} = K Z^{\eta/\delta} \qquad \ldots(9)$$

where $\eta = b_2 a_3 - b_3 a_2$, K is constant.

On equating the expression for N_1 from equation (6) and (8)

$$\frac{1}{b_2}\left[\delta Z \frac{d}{dZ}\{\log N_2\} + a_2\right] = \frac{1}{b_2}\left[\delta Z \frac{d}{dZ}[\log N_4(Z)] + a_4\right]$$

Solving the equation

$$N_2^{b_2} N_4^{-b_4} = L Z^{\sigma/\delta} \qquad \ldots(10)$$

where $\sigma = b_2 a_4 - a_2 b_4$, L is constant

CASE – 1

$\eta = \quad b_2 a_3 - a_2 b_3 > 0$

$\sigma = \quad b_2 a_4 - a_2 b_4 > 0$

Rewriting the equation (9) and (10)

$$N_2^{b_2} N_3^{-b_3} = K Z^{\eta/\delta}$$

$$N_2^{b_2} N_4^{-b_4} = L Z^{\sigma/\delta}$$

For $\eta > 0$, $\sigma > 0$, right hand side of above equation goes to ∞ as $Z \to \infty$.

Hence in equation 8(b) the constant term c_0 and d_0 of the N_3 and N_4 will be zero.

$$N_1(Z) = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-1} \qquad \text{...(11)}$$

$$N_2(Z) = b_0 + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \qquad \text{...(12)}$$

$$N_3(Z) = \sum_{n=1}^{\infty} c^{-n} Z^{-n} \qquad \text{...(13)}$$

$$N_4(Z) = \sum_{n=1}^{\infty} d_{-n} Z^{-n} \qquad \text{...(14)}$$

Substituting the equation (11), (12), (13) and (14) in equation (5), (6), (7) and (8) and comparing coefficients of like powers of z.

$$a_0 = \frac{a_2}{b_2}$$

$$b_0 = \frac{b_2 a_1 - b_1 a_2}{b_2 c_1}$$

Hence

$$\lim_{t\to\infty} N_1(Z) = \frac{a_2}{b_2}$$

$$\lim_{t\to\infty} N_2(Z) = \frac{b_2 a_1 - b_1 a_2}{b_2 c_1}$$

$$\lim_{t\to\infty} N_3(Z) = 0$$

$\lim_{t\to\infty} N_4(Z) = 0$

CASE – 2

$(\eta < 0, \sigma < 0)$

Re- writing the equation (9) and (10)

$$N_2^{b_2} N_3^{-b_3} = KZ^{\eta/\delta} \qquad \text{...(9a)}$$

$$N_2^{b_2} N_4^{-b_4} = L\, Z^{\sigma/\delta} \qquad \text{...(10 a)}$$

For $\sigma < 0$, $\eta < 0$, right hand side of above equation is zero as $Z \to \infty$ .So the constant term b_0 in N_2 must be zero.

$$N_1(Z) = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}$$

$$N_2(Z) = \sum_{n=1}^{\infty} b_{-n} Z^{-n}$$

$$N_3(Z) = c_0 + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \qquad \text{...(15)}$$

$$N_4(Z) = d_0 + \sum_{n=1}^{\infty} d_{-n} Z^{-n}$$

On substituting (15) in equations (5), (6), (7) and (8). We get the constant term.

$$a_0 = \frac{a_3}{b_3} = \frac{a_4}{b_4}$$

$$c_0 d_1 + d_0 e_1 = a_1 - b_1 a_0$$

Hence the solution:

$$\lim_{t\to\infty} N_1 = \frac{a_3}{b_3} = \frac{a_4}{b_4}$$

$$\lim_{t\to\infty} N_2 = 0$$

$$\lim_{t\to\infty} N_3 = c_0$$

$$\lim_{t\to\infty} N_4 = d_0$$

where $\rightarrow C_0\, d_1 + d_0\, e_1 = a_1 - a_0\, b_1$

CASE – III

$(\eta = 0,\ \sigma < 0)$

Rewriting the equations (9) and (10)

$$N_2^{\,b_2} N_3^{\,-b_3} = K\,Z^{\eta/\delta} \qquad \text{- 9a}$$

$$N_2^{\,b_2} N_4^{\,-b_4} = L\;Z^{\sigma/\delta} \qquad \text{-10 a}$$

For $\eta > 0$, right hand side of (9 a) goes to ∞ as Z goes to ∞.

Hence constant term in N_3 (c_0) will be zero.

Since $\sigma < 0$, right hand side of (10 a) goes to zero as Z goes to ∞.

Hence constant term b_0 in N_2 will be zero.

Suppose

$$N_2 = \frac{b_{-1}}{Z}$$

$$N_3 = \frac{c_{-1}}{Z}$$

Putting these in equation (9 a)

$$\left(\frac{b_{-1}}{Z}\right)^{b_2}\left(\frac{c_{-1}}{Z}\right)^{-b_3} = L\, Z^{\eta/\delta}$$

$$c_{-1}^{\,b_2}\, c_{-1}^{\,-b_3}\, Z^{b_3-b_2} = L\, Z^{\eta/\delta}.$$

Since $\eta > 0$, So $b_3 - b_2 > 0$

New conditions are

$$\eta > 0, \sigma < 0, \quad \text{and } b_3 - b_2 > 0$$

So,

$$N_1 = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}$$

$$N_2 = \frac{b_{-1}}{Z} \qquad \ldots(17)$$

$$N_3 = \frac{c_{-1}}{Z}$$

$$N_4 = d_0 + \sum_{n=1}^{\infty} d_{-n} Z^{-n}$$

On substituting the (17) in equations (5, (6), (7) and (8) and comparing coefficients

$$a_0 = \frac{a_4}{b_4}$$

$$a_1 - b_1 a_0 = e_1 d_0.$$

$$d_0 = \frac{a_1 b_4 - b_1 a_4}{b_4 e_1}$$

Hence the solution

$$\lim_{Z \to \infty} N_1 = a_0 = \frac{a_4}{b_4}$$

$$\lim_{Z \to \infty} N_2 = 0$$

$$\lim_{Z \to \infty} N_3 = 0$$

$$\lim_{Z \to \infty} N_4 = \frac{a_1 b_4 - b_1 a_4}{b_4 e_1}$$

CASE – IV

$(\eta < 0, \quad \sigma > 0)$

Rewriting the equation (9) and (10)

$$N_2^{\,b_2} N_3^{\,-b_3} = K\ Z^{\eta/\delta} \qquad \ldots(9\text{ a})$$

$$N_2^{\,b_2}\ N_4^{\,-b_4} = K\ Z^{\sigma/\delta} \qquad \ldots(10\text{ a})$$

For $\eta < 0$, right hand side of (9 a) goes to zero as Z goes to ∞.

Hence constant term b_0 in N_2 will be zero.

Since $\sigma > 0$, right hand side of (10 a) goes to infinite as Z goes to ∞.

Hence constant term d_0 in N_4 will be zero.

Let $\qquad N_1 = a_0 + \sum_{n=1}^{\infty} a_{-n} Z^{-n}.$

$$\begin{cases} N_2 = \dfrac{b_{-1}}{Z} \\ N_3 = c_0 + \sum\limits_{n=1}^{\infty} c_{-n} Z^{-n} \\ N_4 = \dfrac{d_{-1}}{Z} \end{cases} \qquad \ldots(18)$$

substituting these in equation (10 a)

$$\left(\frac{b_{-1}}{Z}\right)^{b_2} \left(\frac{d_{-1}}{Z}\right)^{-b_4} = L\, Z^{\sigma/\delta}.$$

$$(b_{-1})^{b_2} (d_{-1})^{-b_4} Z^{b_4 - b_2} = L\, Z^{\sigma/\delta}.$$

Since $\sigma > 0$, hence $b_4 - b_2 > 0$.

New conditions are

$\eta < 0, \sigma > 0$ and $b_4 - b_2 > 0$

Substituting equation (18) in equations (5), (6), (7) and (8), and comparing coefficients of like powers of Z.

$$a_0 = \frac{a_3}{b_3}$$

$$a_1 - b_1 \; a_0 = d_1 \, c_0.$$

or $$c_0 = \frac{a_1 b_3 - b_1 a_3}{d_1 b_3}$$

Hence

$$\lim_{t \to \infty} N_1 = a_0 = \frac{a_3}{b_3}$$

$$\lim_{t \to \infty} N_2 = 0$$

$$\lim_{t \to \infty} N_3 = \frac{a_1 b_3 - b_1 a_3}{d_1 b_3}$$

$$\lim_{t \to \infty} N_4 = 0$$

CHAPTER - 4

ILLUSTRATION OF THE ANALYTICAL RESULTS USING RUNGE-KUTTA APPROXIMATION METHOD

In this chapter we illustrate our previously obtained results using the Rung-Kutta approximation method for numerical analysis. The program used for this purpose is a standard Runge Kutta Fourth order. We fed our specific numerical inputs in the program and the results under different conditions were plotted.

RESULTS

ONE PREY-THREE PREDATOR SYSTEM

Case 1: For $\sigma > 0$ i.e $b_2 a_3 - a_2 b_3 > 0$

$\eta > 0$ i.e. $b_3 a_4 - a_2 b_4 > 0$

Initial value of the populations:

$N_1(0) = 2$

$N_2(0) = 3$

$N_3(0) = 2$

$N_4(0) = 3$

Numerical inputs for Different Parameters:

$a_1 = 2$	$b_2 = 8$
$b_1 = 0.5$	$a_3 = 3$
$c_1 = 0.2$	$b_3 = 1.5$
$d_1 = 0.8$	$a_4 = 4$

$e_1 = 8$ $b_4 = 2$

$a_2 = 8$

The situation for this case is represented by Fig. 1.

Case 2: For $\sigma < 0$

$\eta < 0$

Initial value of the populations

$N_1(0) = 3$

$N_2(0) = 4$

$N_3(0) = 1$

$N_4(0) = 4$

Numerical inputs for different parameters:

$a_1 = 2$ $b_2 = 3$

$b_1 = 0.5$ $a_3 = 3$

$c_1 = 0.2$ $b_3 = 1.5$

$d_1 = 0.8$ $a_4 = 4$

$e_1 = 0.1$ $b_4 = 2$

$d_2 = 8$

The Situation for this case is represented by Fig. 2.

Case 3: For $\sigma > 0$

$\eta < 0$

Initial value of the populations:

$N_1(0) = 3$

$N_2(0) = 4$

$N_3(0) = 4$

$N_4(0) = 3$

Numerical inputs for different parameters:

$a_1 = 2$	$b_2 = 3$
$b_1 = .5$	$a_3 = 3$
$c_1 = .2$	$b_3 = 1.5$
$d_1 = .8$	$a_4 = 11$
$e_1 = .1$	$b_4 = 4$
$d_2 = 8$	

Situation for this case is represented by Fig. 3.

Case 4: For $\sigma < 0$

$\eta > 0$

Initial value of the populations:

$N_1(0) = 2$

$N_2(0) = 4$

$N_3(0) = 6$

$N_4(0) = 8$

Numerical inputs for different parameters:

$a_1 = 2.5$	$b_2 = 8$
$b_1 = 0.5$	$a_3 = 10$
$c_1 = 0.2$	$b_3 = 9$
$d_1 = 0.8$	$a_4 = 4$
$e_1 = 0.1$	$b_4 = 5$

$a_2 = 8$

The situation for this case is represented in Fig. 4.

THREE PREY-ONE PREDATOR SYSTEM

Case 1: For $\sigma > 0$ i.e $a_1 b_2 - b_1 a_2 > 0$

$\eta > 0$ i.e. $a_1 b_3 - b_1 a_3 > 0$

Initial value of the populations:

$N_1(0) = 5$

$N_2(0) = 5$

$N_3(0) = 6$

$N_4(0) = 2$

Numerical inputs for different parameters:

$a_1 = 4$	$a_4 = 2$
$b_1 = 2$	$b_4 = 0.5$
$a_2 = 3$	$c_4 = 0.2$
$b_2 = 3$	$d_4 = 0.8$
$a_3 = 3$	$e_4 = 0.1$
$b_3 = 3$	

The situation for this case is represented by Fig. 5.

Case 2: For $\sigma < 0$

$\eta < 0$

Initial value of the populations:

$N_1(0) = 5$

$N_2(0) = 10$

$N_3(0) = 5$

$N_4(0) = 5$

Numerical inputs for different parameters:

$a_1 = 3$	$a_4 = 2$
$b_1 = 4$	$b_4 = 0.5$
$a_2 = 6$	$c_4 = 0.2$
$b_2 = 3$	$d_4 = 0.8$
$a_3 = 6$	$e_4 = 0.1$
$b_3 = 3$	

The situation for this case is represented by Fig. 6.

Case 3: For $\sigma > 0$

$\eta < 0$

Initial value of populations

$N_1(0) = 3$

$N_2(0) = 5$

$N_3(0) = 4$

$N_4(0) = 2$

Numerical inputs for different parameters:

$a_1 = 3$	$a_4 = 2$
$b_1 = 4$	$b_4 = 0.5$
$a_2 = 2$	$c_4 = 0.2$
$b_2 = 3$	$d_4 = 0.8$
$a_3 = 3$	$e_4 = 0.3$

$b_3 = 4$

The situation for this case is represented by Fig. 7.

Case 4: For $\sigma < 0$

$\eta > 0$

Initial value of populations:

$N_1(0) = 6$

$N_2(0) = 5$

$N_3(0) = 8$

$N_4(0) = 2$

Numerical inputs for different parameters:

$a_1 = 3$	$a_4 = 2$
$b_1 = 6$	$b_4 = 0.5$
$a_2 = 3$	$c_4 = 0.2$
$b_2 = 3$	$d_4 = 0.8$
$a_3 = 2$	$e_4 = 0.1$
$b_3 = 5$	

The situation for this case is represented by Fig. 8.

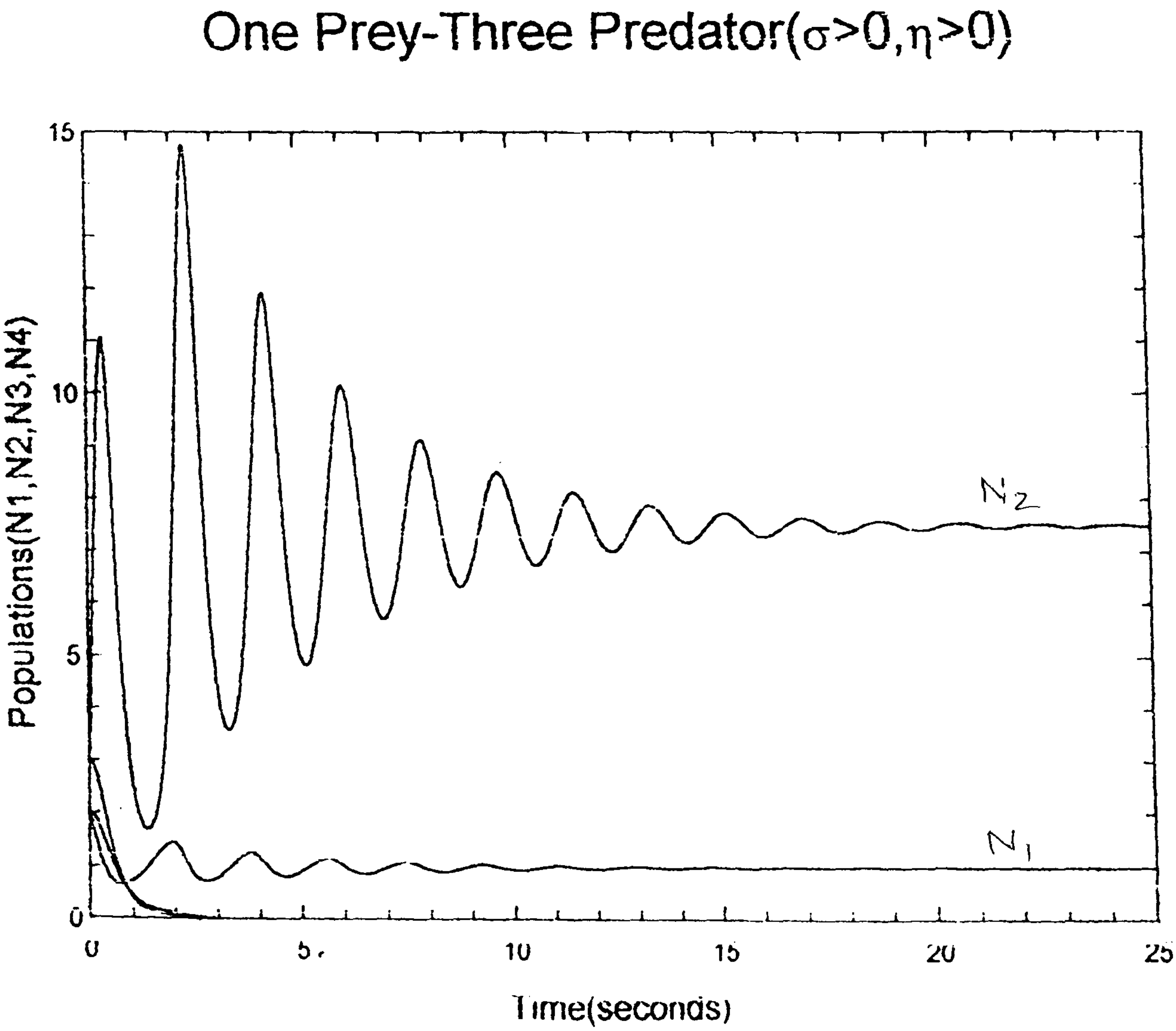

One Prey-Three Predator(σ>0,η>0)
15
10
5
0
0
5
10
15
20
25
Populations(N1,N2,N3,N4)
Time(seconds)
N2
N1

FIG. 1

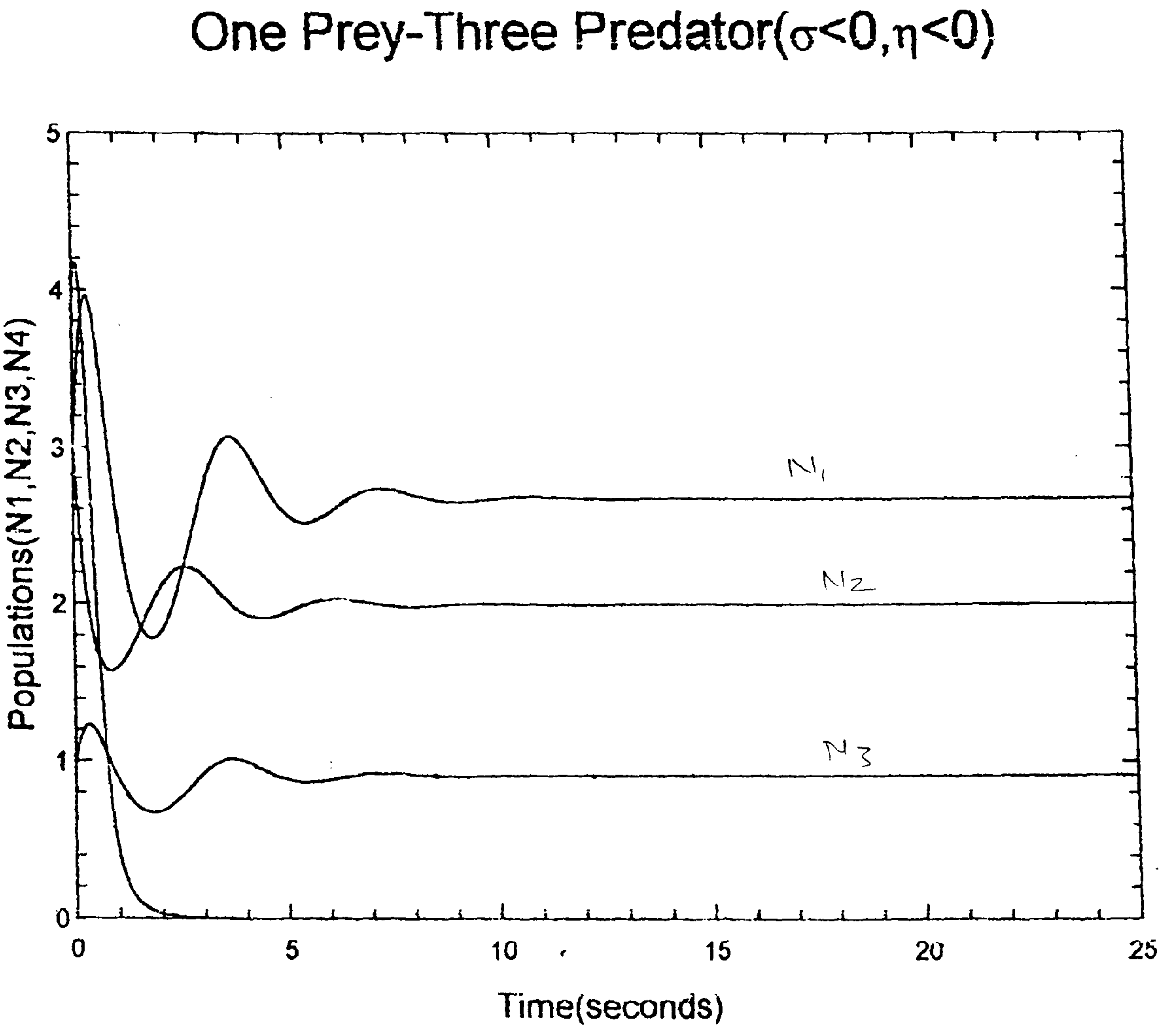

One Prey-Three Predator(σ<0,η<0)
5
4
3
2
1
0
Populations(N1,N2,N3,N4)
0
5
10
15
20
25
Time(seconds)
N_1
N_2
N_3

FIG 2

One Prey-Three Predator($\sigma>0,\eta<0,(b4-b2)>0$)

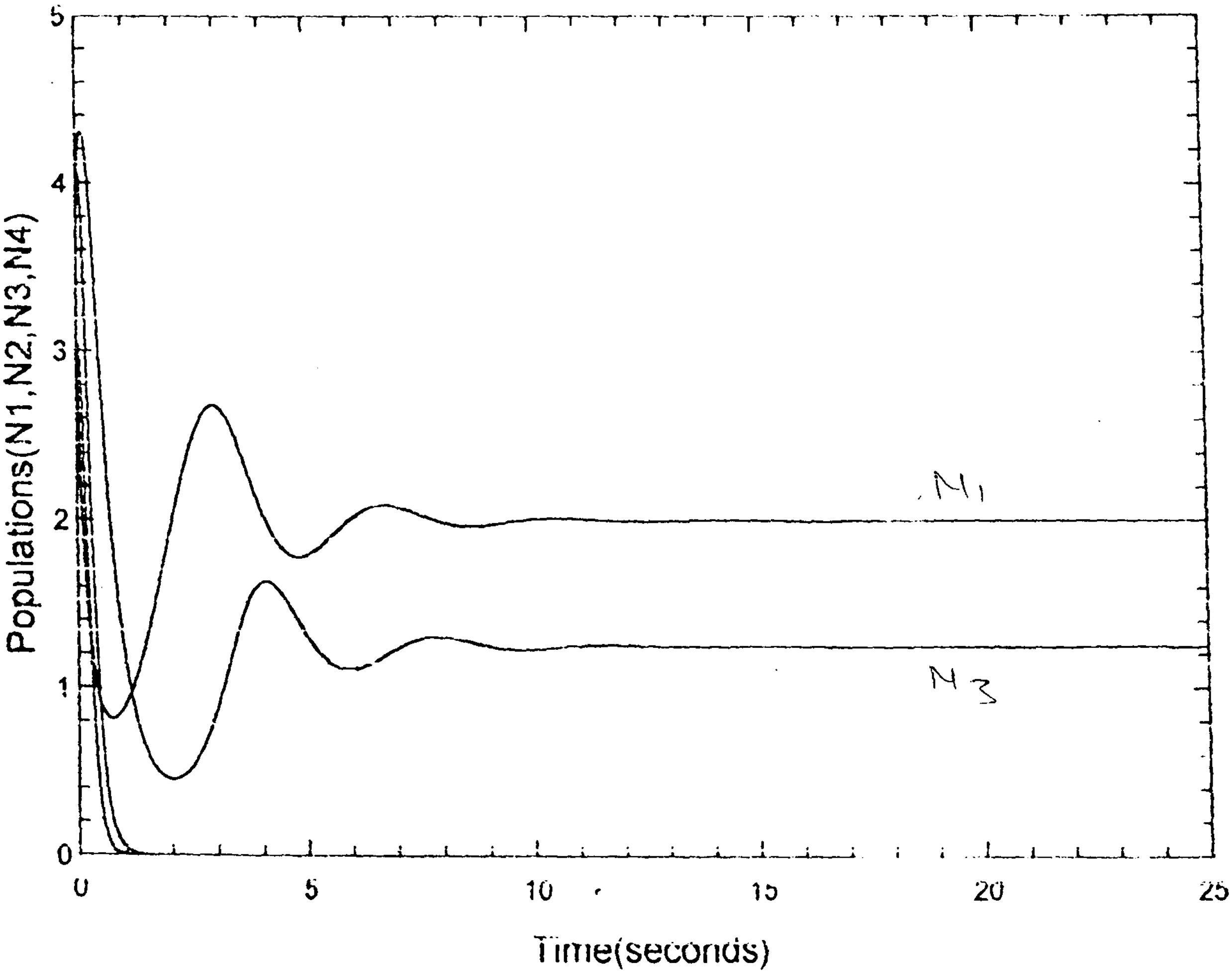


FIG. 3

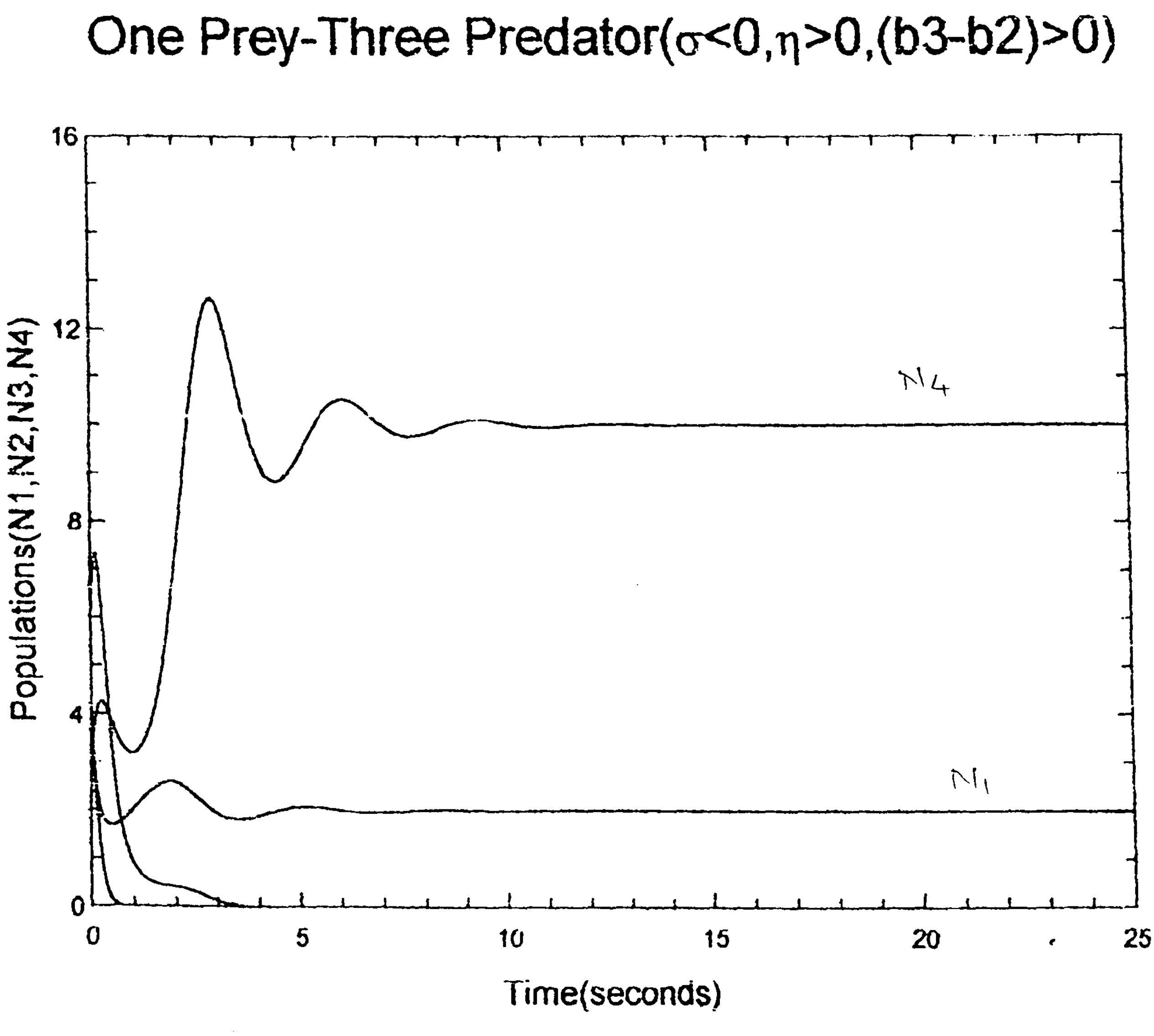

One Prey-Three Predator(σ<0,η>0,(b3-b2)>0)
Populations(N1,N2,N3,N4)
Time(seconds)
N4
N1
0
4
8
12
16
0
5
10
15
20
25

FIG. 4

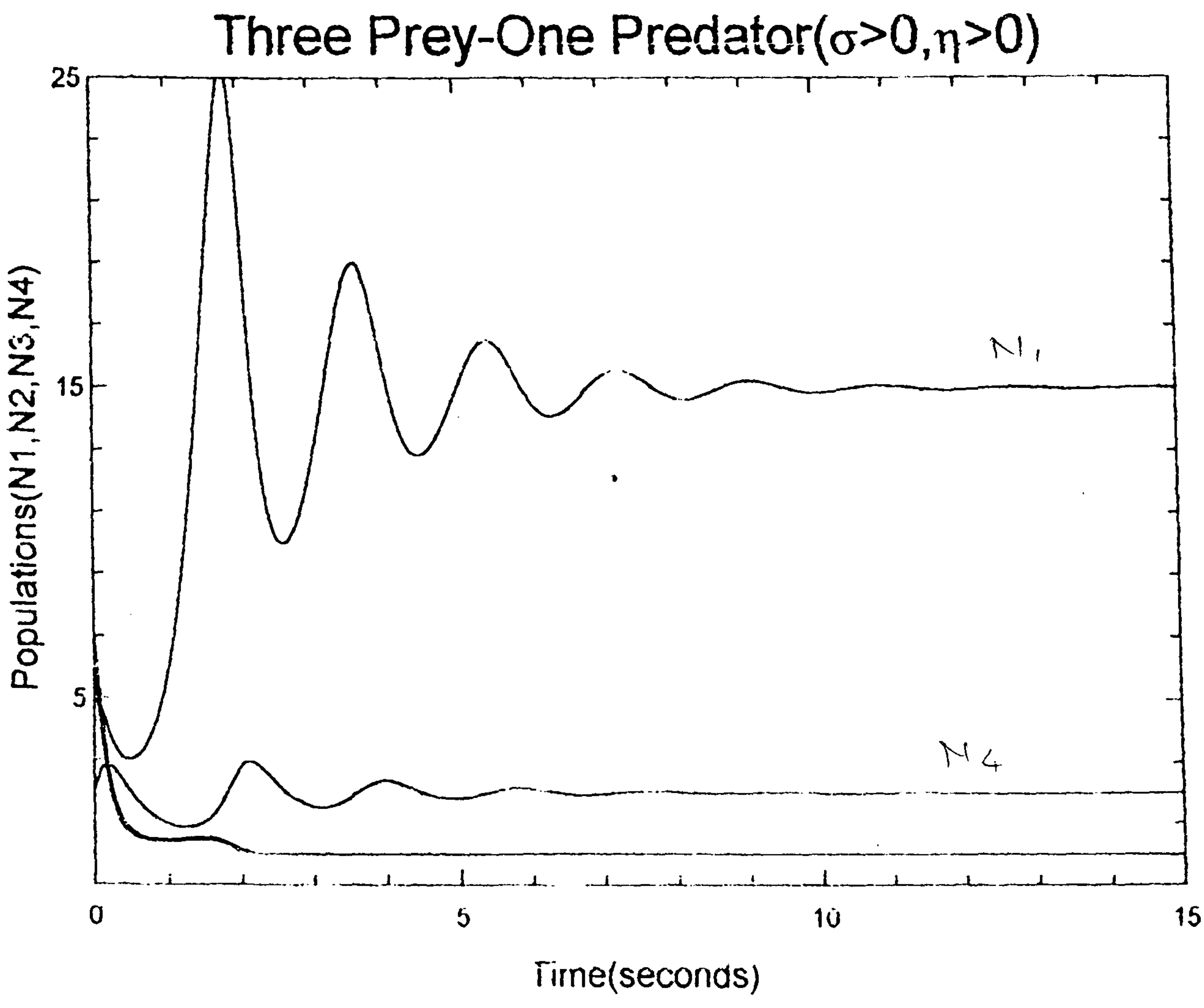

Three Prey-One Predator(σ>0,η>0)
25
15
5
Populations(N1,N2,N3,N4)
N1
N4
0
5
10
15
Time(seconds)

FIG - 5

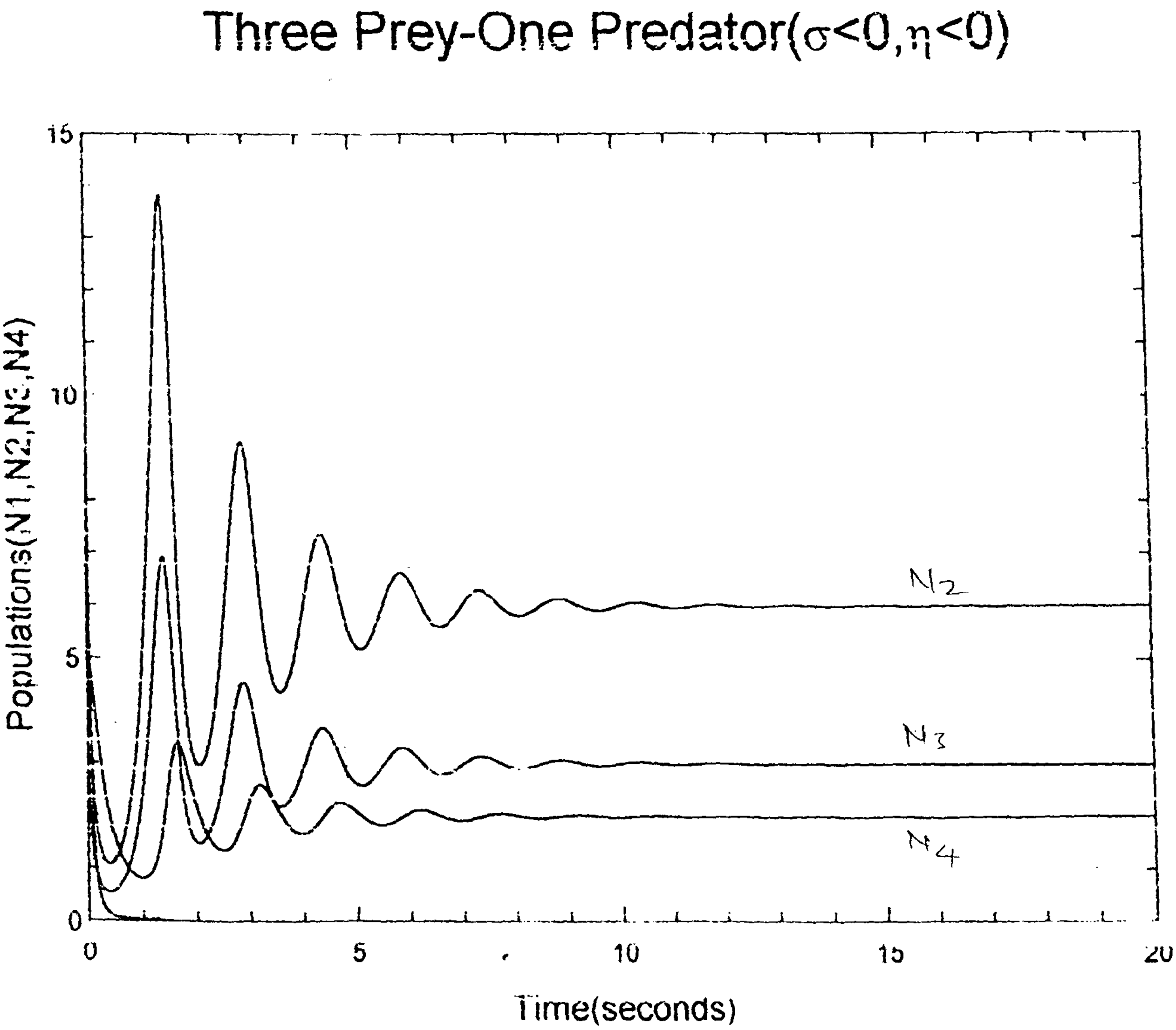
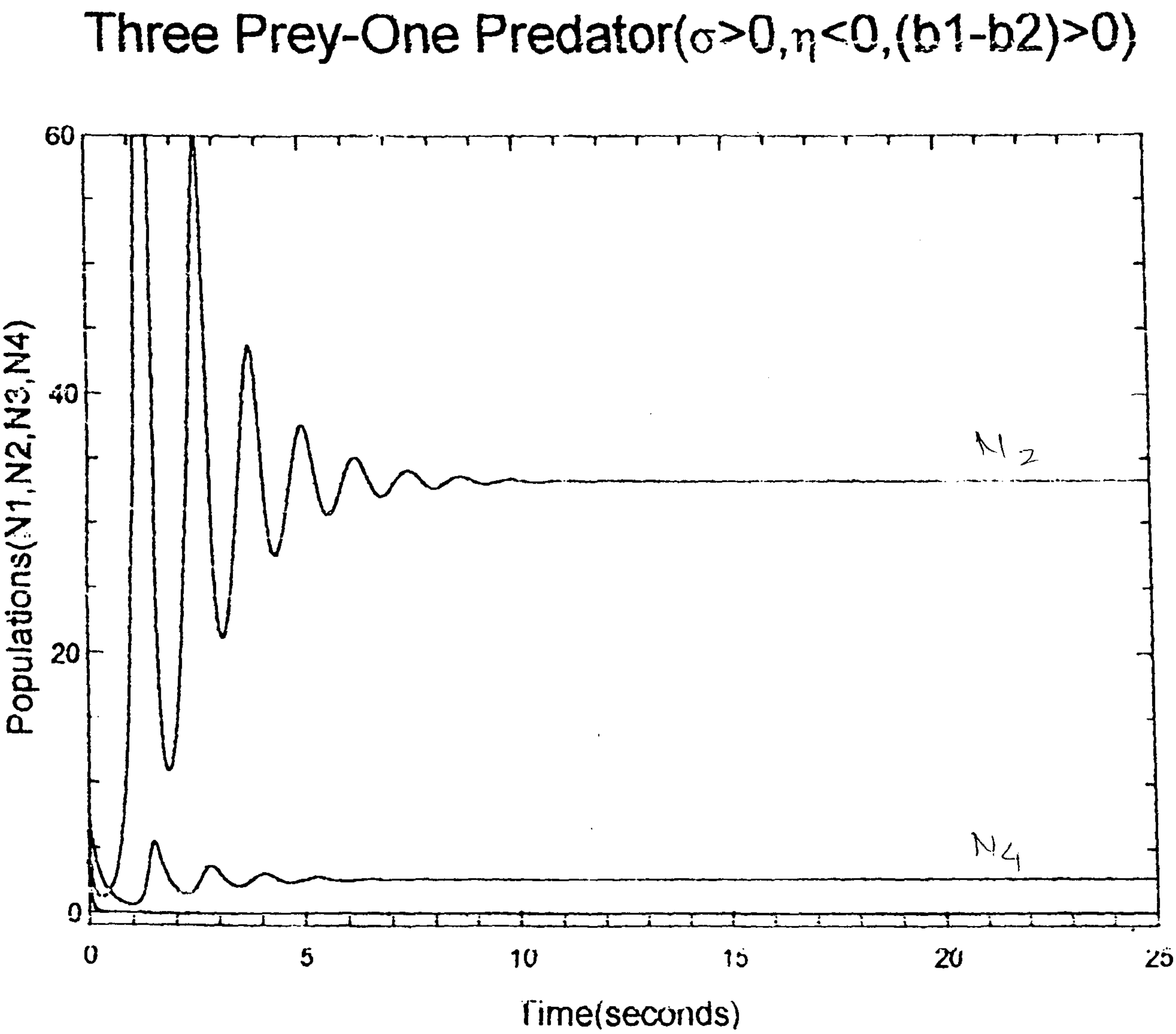
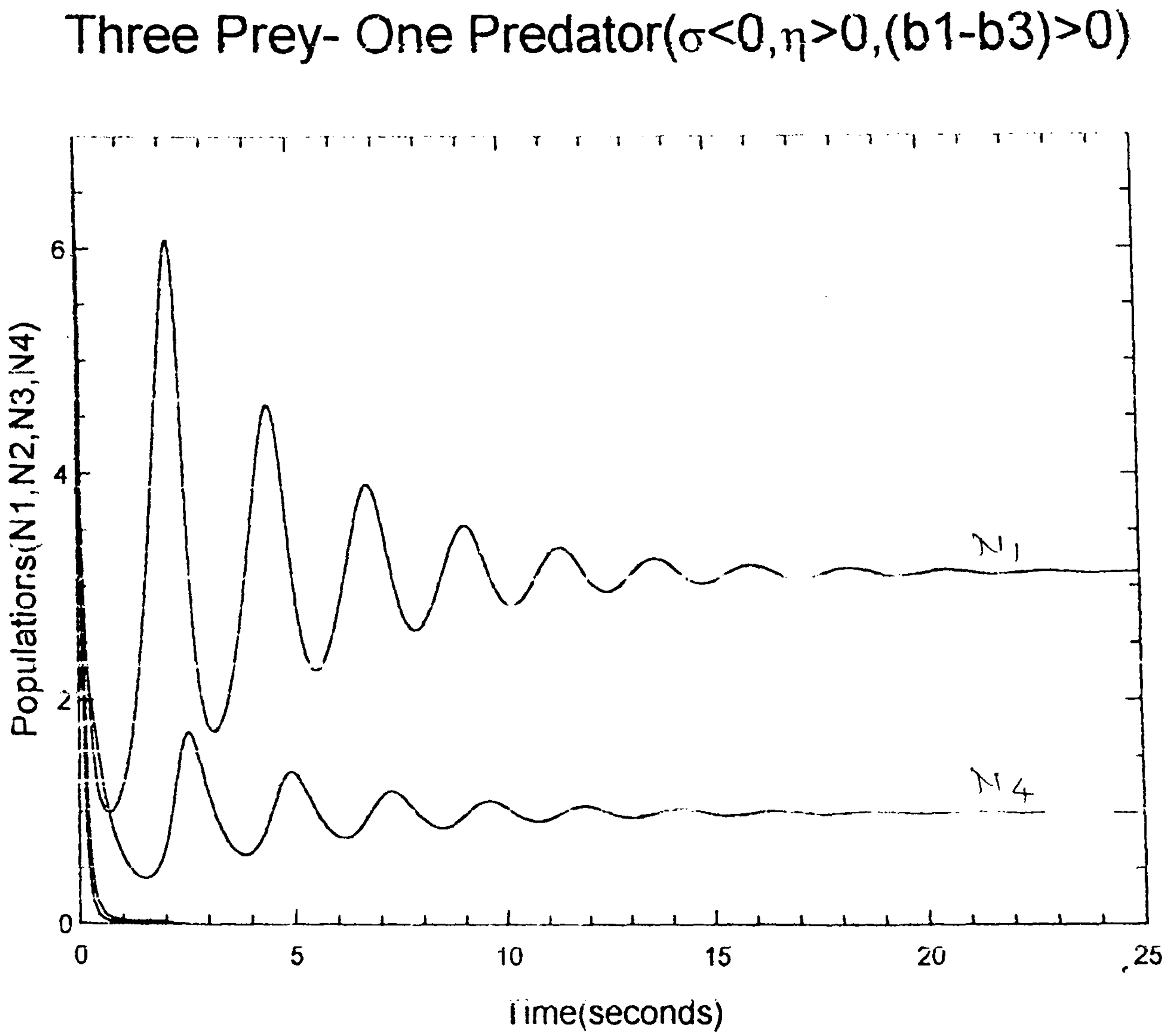

Three Prey-One Predator(σ<0,η<0)
15
10
5
0
0
5
10
15
20
Time(seconds)
Populations(N1,N2,N3,N4)
N2
N3
N4

FIG. 6



Three Prey-One Predator(σ>0,η<0,(b1-b2)>0)
60
40
20
0
Populations(N1,N2,N3,N4)
N2
N4
0
5
10
15
20
25
Time(seconds)

FIG. - 7



Three Prey- One Predator(σ<0,η>0,(b1-b3)>0)
Populations(N1,N2,N3,N4)
Time(seconds)
0
2
4
6
0
5
10
15
20
25
N1
N4

FIG. 8

CHAPTER - 5

CONCLUSION

We have obtained the asymptotic behavior of the component population for two different four species models. This has been done by exploiting constraints which exist in the sub space of three of the four species in each case, and by using a Laurent expansion in suitably chosen variable. Our main results are summarized in the following tables.

ASSYMPTOTIC BEHAVIOUR OF POPULATIONS IN THREE PREY ONE PREDATOR MODEL

MODEL	BEHAVIOUR FOR $t \to \infty$.
$\dot{N}_1 = a_1 N_1 - b_1 N_1 N_4$	**CASE – 1**
$\dot{N}_2 = a_2 N_2 - b_2 N_2 N_4$	$\sigma > 0, \eta > 0$
$\dot{N}_3 = a_3 N_3 - b_3 N_3 N_4$	$N_1 = \dfrac{a_4 b_1 + a_1 b_4}{c_4 b_1}$
$\dot{N}_4 = -a_4 N_4 - b_4 N_4^2 + C_4 N_1 N_4$	$N_1 = 0$
$+ d_4 N_2 N_4 + C_4 N_3 N_4$	$N_3 = 0$
	$N_4 = \dfrac{a_1}{b_1}$
Constraints	**CASE – 2**
i. $N_1^{b_2} N_2^{-b_1} = K Z^{\sigma/\delta}$.	$\sigma < 0, \eta < 0$
Where $\sigma = a_1 b_2 - b_1 a_2$.	$N_1 = 0$

$N_1^{b_3} \; N_3^{-b_1} = L\, Z^{\eta/\delta}.$

$\eta = a_1 b_3 - b_1 a_3.$

$N_2 = b_0$

$N_3 = C_0$

$$N_4 = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Where $a_4 + b_4 d_0 = d_4 b_0 + e_4 c_0$

CASE III

$\sigma > 0, \eta < 0$

$N_1 = 0$

$N_2 = 0$

$$N_3 = \frac{a_4 b_3 + a_3 b_4}{b_3 e_4}$$

$$N_4 = d_0 = \frac{a_3}{b_3}$$

CASE – IV

$(\sigma < 0, \quad \eta > 0)$

$N_1 = 0$

$$N_2 = \frac{a_4 b_2 + a_2 b_4}{b_2 d_4}$$

$N_3 = 0$

$$N_4 = \frac{a_2}{b_2}$$

ASSYMPTOTIC BEHAVIOUR OF POPULATIONS IN ONE PREY – THREE PREDATOR MODEL

MODEL	**BEHAVIOUR FOR $t \to \infty$.**
$\dot{N}_1 = a_1 N_1 - b_1 N_1^2 + C_1 N_1 N_2$	**CASE – 1** ($\sigma > 0, \eta > 0$)
$d_1 N_1 N_3 + e N_1 N_4$	$N_1 = \dfrac{a_2}{b_2}$
$\dot{N}_2 = -a_2 N_2 + b_2 N_2 N_4$	$N_2 = \dfrac{b_2 a_1 - b_1 a_2}{b_2 C_1}$
$\dot{N}_3 = -a_3 N_3 + b_3 N_1 N_3$	$N_3 = 0$
$\dot{N}_4 = -a_4 N_4 + b_4 N_1 N_4$	$N_4 = 0$
Constraints	**CASE – 2**
	($\sigma < 0, \eta < 0$)
$N_2^{b_2} N_4^{-b_4} = L Z^{\sigma/\delta}$	$N_1 = \dfrac{a_3}{b_3} = \dfrac{a_4}{b_4}$
$N_2^{b_2} N_3^{-b_3} = K Z^{\eta/\delta}$	$N_2 = 0$
	$N_3 = c_0$
where	$N_4 = d_0$
$\sigma = b_2 a_4 - a_2 b_4$	Where $c_0 d_1 + d_0 e_1 = a_1 - b_1 a_0$
$\eta = b_2 a_3 - b_3 a_2$.	**CASE – 3**
	($\sigma < 0, \eta > 0$)
	$b_3 - b_2 > 0$

$$N_1 = \frac{a_4}{b_4}$$

$$N_2 = 0$$

$$N_3 = 0$$

$$N_4 = \frac{a_1 b_4 - b_1 a_4}{b_4 e_1}$$

CASE – 4

($\sigma > 0, \eta < 0.$)

$$N_1 = \frac{a_3}{b_3}$$

$$N_2 = 0$$

$$N_3 = \frac{a_1 b_3 - b_1 a_3}{d_1 b_3}$$

$$N_4 = 0$$

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```
DECLARE SUB lor (x!(), nn!, f!())
SCREEN 9
CLS
nn = 4
DIM w(nn), x0(nn), x1(nn), x2(nn), x3(nn), x4(nn), f(nn), x(nn)
'OPEN "data2" FOR OUTPUT AS #1
READ hh, tin, tend, hprint
DATA .005,0,30,.1
FOR i = 1 TO nn
READ x0(i)
NEXT
DATA 2,4,6,8
aa = 2.5
bb = .5
cc = .2
dd = .8
ee = .1
ff = 8
gg = 8
ii = 10
jj = 9
kk = 4
ll = 5
j = 0
10 'graphics
 VIEW
 WINDOW (0, -25)-(tend, 25)
' WINDOW (-2, -2)-(2, 2)
REM LINE (-15, 0)-(tend, 0)
LINE (-15, -25)-(-15, 25)
FOR i = 1 TO nn
w(i) = x0(i)
NEXT
t = tin
250 'output from subprogram comes here
'PRINT t; w(1), w(2), w(3)
j = j + 1
'plotting done here
tp = t
xp = w(1)
yp = w(2)
zp = w(3)
up = w(4)
IF j = 1 THEN 11
'WRITE #1, tp, xp, yp, zp, up
REM LINE (told, xo)-(tp, xp)
PSET (tp, xp)
REM LINE (told, yo)-(tp, yp)
PSET (tp, yp)
REM LINE (told, zo)-(tp, zp)
PSET (tp, zp)
REM LINE (told, uo)-(tp, up)
PSET (tp, up)
'LINE (xo, yo)-(xp, yp)
11 told = tp
   xo = xp
   yo = yp
   zo = zp
   uo = up
IF t < tend THEN GOTO 555
'CLOSE (1)
STOP
555
h2 = .5 * hh
```

```
460 FOR i = 1 TO nn
x0(i) = w(i)
NEXT
CALL lor(x0(), nn, f())
FOR i = 1 TO nn
x1(i) = hh * f(i)
NEXT
t = t + h2
FOR i = 1 TO nn
x(i) = w(i) + x1(i) * .5
NEXT
CALL lor(x(), nn, f())
FOR i = 1 TO nn
x2(i) = hh * f(i)
NEXT
FOR i = 1 TO nn
x(i) = w(i) + x2(i) * .5
NEXT
CALL lor(x(), nn, f())
FOR i = 1 TO nn
x3(i) = hh * f(i)
NEXT
FOR i = 1 TO nn
x(i) = w(i) + x3(i)
NEXT
t = t + h2
CALL lor(x(), nn, f())
FOR i = 1 TO nn
x4(i) = hh * f(i)
NEXT
FOR i = 1 TO nn
x(i) = w(i) + (x1(i) + 2 * x2(i) + 2 * x3(i) + x4(i)) / 6
NEXT
 FOR i = 1 TO nn
 w(i) = x(i)
 NEXT
1200 GOTO 250
STOP
END


SUB lor (x(), nn, f()) STATIC
SHARED aa, bb, cc, dd, ee, ff, gg, ii, jj, kk, ll
f(1) = aa * x(1) - bb * x(4) * x(1)
f(2) = cc * x(2) - dd * x(2) * x(4)
f(3) = ee * x(3) - ff * x(3) * x(4)


f(4) = -gg * x(4) - ii * x(4) * x(4) + jj * x(1) * x(4) + kk * x(2) * x(4) + l
END SUB
```

```
DECLARE SUB lor (x!(), nn!, f!())
SCREEN 9
CLS
nn = 4
DIM w(nn), x0(nn), x1(nn), x2(nn), x3(nn), x4(nn), f(nn), x(nn)
'OPEN "data1" FOR OUTPUT AS #1
READ hh, tin, tend, hprint
DATA .005,0,30,.1
FOR i = 1 TO nn
READ x0(i)
NEXT
DATA 3,5,8,10
aa = 3
bb = 4
cc = 2
dd = 3
ee = 3
ff = 3
gg = 2
ii = .5
jj = .2
kk = .8
ll = .1
j = 0
10 'graphics
 VIEW
 WINDOW (0, -25)-(tend, 50)
'WINDOW (-2, -2)-(2, 2)
REM LINE (-15, 0)-(tend, 0)
LINE (-15, -25)-(-15, 50)
FOR i = 1 TO nn
w(i) = x0(i)
NEXT
t = tin
250 'output from subprogram comes here
'PRINT t; w(1), w(2), w(3)
j = j + 1
'plotting done here
tp = t
xp = w(1)
yp = w(2)
zp = w(3)
up = w(4)
IF j = 1 THEN 11
'WRITE #1, tp, xp, yp, zp, up
REM LINE (told, xo)-(tp, xp)
PSET (tp, xp)
REM LINE (told, yo)-(tp, yp)
PSET (tp, yp)
REM LINE (told, zo)-(tp, zp)
PSET (tp, zp)
REM LINE (told, uo)-(tp, up)
PSET (tp, up)
'LINE (xo, yo)-(xp, yp)
11 told = tp
   xo = xp
   yo = yp
   zo = zp
   uo = up
IF t < tend THEN GOTO 555
'CLOSE (1)
STOP
555
h2 = .5 * hh
```

```
460 FOR i = 1 TO nn
x0(i) = w(i)
NEXT
CALL lor(x0(), nn, f())
FOR i = 1 TO nn
x1(i) = hh * f(i)
NEXT
t = t + h2
FOR i = 1 TO nn
x(i) = w(i) + x1(i) * .5
NEXT
CALL lor(x(), nn, f())
FOR i = 1 TO nn
x2(i) = hh * f(i)
NEXT
FOR i = 1 TO nn
x(i) = w(i) + x2(i) * .5
NEXT
CALL lor(x(), nn, f())
FOR i = 1 TO nn
x3(i) = hh * f(i)
NEXT
FOR i = 1 TO nn
x(i) = w(i) + x3(i)
NEXT
t = t + h2
CALL lor(x(), nn, f())
FOR i = 1 TO nn
x4(i) = hh * f(i)
NEXT
FOR i = 1 TO nn
x(i) = w(i) + (x1(i) + 2 * x2(i) + 2 * x3(i) + x4(i)) / 6
NEXT
 FOR i = 1 TO nn
 w(i) = x(i)
 NEXT
1200 GOTO 250
STOP
END


SUB lor (x(), nn, f()) STATIC
SHARED aa, bb, cc, dd, ee, ff, gg, ii, jj, kk, ll
f(1) = aa * x(1) - bb * x(1) * x(1) - cc * x(1) * x(2) - dd * x(1) * x(3) - ee
f(2) = -ff * x(2) + gg * x(1) * x(2)
f(3) = -ii * x(3) + jj * x(1) * x(3)
f(4) = -kk * x(4) + ll * x(1) * x(4)
END SUB
```