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MODELLING OF TROPICAL CYCLONES

Dissertation submitted to Jawaharlal Nehru University
in partial fulfilment of the requirements
for the award of the Degree of
MASTER OF PHILOSOPHY

53p.

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JAWAHARLAL NEHRU UNIVERSITY
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1993

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CERTIFICATE

Certified that the work embodied in this dissertation entitled "Modelling of Tropical cyclones" has been carried out in the School of Environmental Sciences, Jawaharlal Nehru University, New Delhi. The work is original and has not been submitted in part or in full for any other degree or diploma in this or in any other University.

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ACKNOWLEDGEMENT

I take this opportune moment to express profound and sincere gratitude to my supervisor Dr. G.P. Malik, without whose guidance this dissertation would n't have been possible. He initiated me in ferreting out critical research results and process of "Scientific writing". His guidance was remarkable for the high degree of professonalism. Infact words can not do full justice to the contribution of Dr. Malik during the various stages of this work.

I am also thankful to Dr. V.K. Jain, who helped me a lot in overcoming a number of hurdles during the whole tenure of my research work. I would be failing in my duty if I were not to acknowledge Prof. J. Subba Rao, Prof. B.P. Murthy and Prof. U.C. Mohanty's help and guidance throughout this dissertation.

Staff of India Meteorology Department need to be thanked for extending me the library facility.

Last but by no means the least, the facilities and financial assistance provided by Dean, SES, J.N.U and University Grants Commission is gratefully acknowledged.

Dharm Prakash

20th July, 1993. (DHARM PRAKASH VAJPEYI)

CONTENTS

Chapter 1

TROPICAL CYCLONES : AN INTRODUCTION

I. Tropical cyclone as an environmental hazard.

A tropical cyclone is a universally recognized environmental agent of destruction. The coastal belts of India, particularly the eastern coastal belt, are extremely vulnerable to its fury. Between 1891-1982, these belts suffered about 300 cyclones[1]. The cyclone that struck the Indian coastal belt on Nov. 13, 1970 caused the death of 200,000 human lives and 800,000 livestock; it destroyed 200,000 houses, 80% of the standing paddy crop, and 65% of the fishing capacity in 9000 localities. Such being the extent of damage that a tropical cyclone can cause, one can scarcely over-emphasize the need to study it on a scientific basis.

II. Observational features

A scientific approach to the study of a tropical cyclone (also called a hurrricane) must begin with its observational features, which we now briefly summarise.

(i) Pre-hurricane features

Palmen[2] has noted that the region where a hurricane might eventually develop is characterized by :

(a) The existence of a low-level disturbance with minimum pressure fall of the order of 25-30 mb;

(b) Sufficiently large sea or ocean areas in the vicinity;

(c) Sea surface temperature above 25°C, higher than ambient temperature;

(d) A value of the Coriolis parameter larger than a certain minimum value (thus excluding a belt of the width of about 5-8° of latitude on either side of the equator;

(e) A high value of specific humidity.

The above conditions are necessary for a hurricane to develop but not sufficient.

(ii) **Features of a mature hurricane**.

Occasionally the conditions described above lead to the growth of a hurricane, which is a nearly circular, warm-cored vortex occupying the entire height of the troposphere and extending radially many hundreds of kilometers. Typical scales associated with a hurricane are :

(a) Region of occurrence : Between 5 and 20° latitudes;

(b) Horizontal scale : 100 km in radius;

(c) Surface pressure gradient : 3mb/km;

(d) Maximum tangential wind speeds : 50-100 m/s;

(e) Rainfall in localized region : 15-25cm in a short duration of time;

(f) Storm-surge height : 3-5 m;

III. **A detailed qualitative model of a hurricane.**[3]

A model of an idealized hurricane structure, which incorporates the above features in a qualitative manner, consists of four dynamically distinct regions. These may be described as follows :

Region I : A boundary layer where the swirl winds interact frictionally with the sea surface, causing an inflow which supplies moist air to the updraft.

Region II : A region of rapidly swirling winds corresponding to the large radial pressure gradient.

Region III : An updraft region of tall convective clouds and intense rainfall with accompanying release of large amounts of latent heat.

Region IV : A warm, quiescent, relatively dry core or eye in which there is a very slow recirculation.

The swirling fluid of region II settles very slowly into the region below, supplying as it does so the fluid which moves radially inward in the boundary layer. There is almost no radial motion at all in II and the flow therein is effectivley frictionless.

Frictional effects are important in the boundary layer, region I, where the swirling flow loses some of its angular momentum to the sea surface. The radial component of the pressure gradient throughout the boundary layer is virtually the same as it is near the bottom of region II, but the frictionally depleted circumferential velocity in the boundary layer does not imply enough centripetal acceleration to balance that pressure gradient in I. Thus, the overall radial momentum balance is maintained in I only, when the friction and momentum changes associated with a radial inflow are present.

The radial influx of fluid in I, whose source is the downdraft from II, moves into the annular region III where it flows upward and outward, conserving angular momentum. At any given radial position the updraft fluid has a deficit in total angular momentum, compared to the fluid in II at the same radius merely because of the frictional loss to the sea surface, which occurred in I. In the outer portions of the storm this deficit will appear as a negative circumferential velocity relative to the earth and in fact this high "anti-cyclonic" motion is always observed.

The fluid in the central core IV is relatively motionless. It is also very warm, so that at a given altitude the fluid is much less dense than the updraft fluid in III, which is itself much less dense than near by fluid in II. These differences in fluid densities in regions II, III, IV imply that a column of air which is closer to the centre than a second column has less total weight than that more distant column, this is necessary conditon for the existence of the horizontal pressure gradient which balances the centripetal acceleration of the swirling flow in region II.

IV. **Towards mathematical modelling of a hurricane**

In order to be able to predict the occurrence and severity of a hurricane, we must set up and solve a set of prognostic equations which adequately describe all the features of the above qualitative model. In particular, this entails taking into account the updrafts of cumulus motion, which is caused by the eddy viscosity of surface winds. This is a major hurdle, since a theory of eddy viscosity does not exist. One is thus forced to parameterize this aspect. Next, we must have a set of equations describing the large-scale motion of the hurricane. These equations must also incorporate in a sensible manner the coupling with the parameterized cumulus scale motion. Equations thus obtained are, in general, nonlinear partial differential equations and non-tractable. However, one may reasonably expect scale-analysis to be of help here-which does turn out to be the case. Finally, the practical way to obtain meaningful information about the hurricane dictates that the equations for conservation of mass, energy and angular momentum be set up and considered along with the momentum equations obtained earlier. These considerations bring out the extremely complex nature of the problem being studied; indeed, the stydy of tropical cyclones is regarded as the most challenging problem in atmospheric sciences.

V. **Scope of this dissertation**

Mathematical modelling of tropical cyclones is a comparatively recent field of study, which originated in 1964 in the work of Charney and Eliassen[4]. This was a pioneering work : not only did it introduce the concept of CISK (Conditional instability of the second kind) which, it proposed, causes the cumulus and large-scale motions to cooperate and grow spontaneously rather than compete for the same energy, but also provided the basic mathematical framework for modelling the hurricane. Indeed, this framework has been of central importance in many models of the hurricane which have been rep~~ea~~orted in the literature in the last 30 years or so.

Because of its importance, the thrust of this dissertation is on a detailed exposition of the Charney-Eliassen model[4]. This is done mostly in chapter 3. In chapter 2, we have attempted to give a self-sufficient and ab-initio account of the equations of dynamic meteorology, familiarity with which is a pre-requisite to understanding the mathematical modelling of the hurricane; this chapter also contains the preliminaries of the Charney-Eliassen model. Finally, in chapter 4, we have attempted to give a selective but uptodate review of the work in the field of mathematical modelling of a hurricane.

REFERENCES

1. A.K. Saha et al., Report of the Cyclone Review Committee, Department of Science and Technology (unpublished), New Delhi, Vol. 2, 480p.
2. E. Palmen and H. Riehl, J. of Met. , 14, 150-159, 1957.
3. G.F. Carrier, A.L. Hammond and O.D. George, J. Flu. Mech., 47, 145-170, 1971.
4. J.G. Charney and A. Eliassen, J. Atmos. Sci., 21, 68-75, 1964.

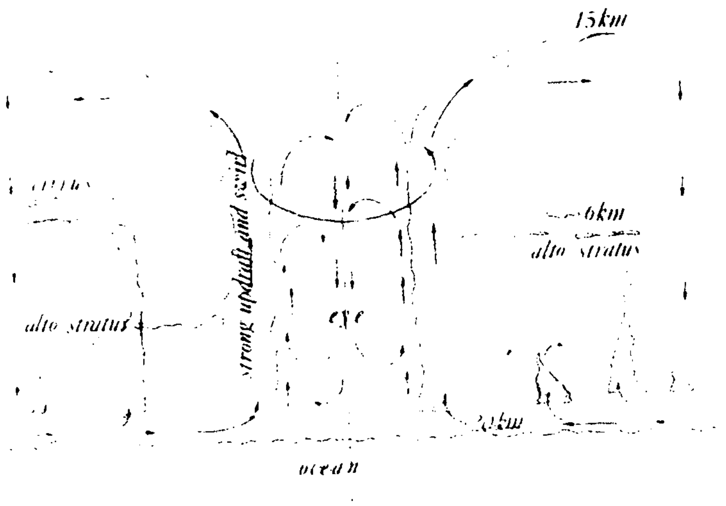

Figure 1 : Vertical Cross-section of a typical Hurricane.

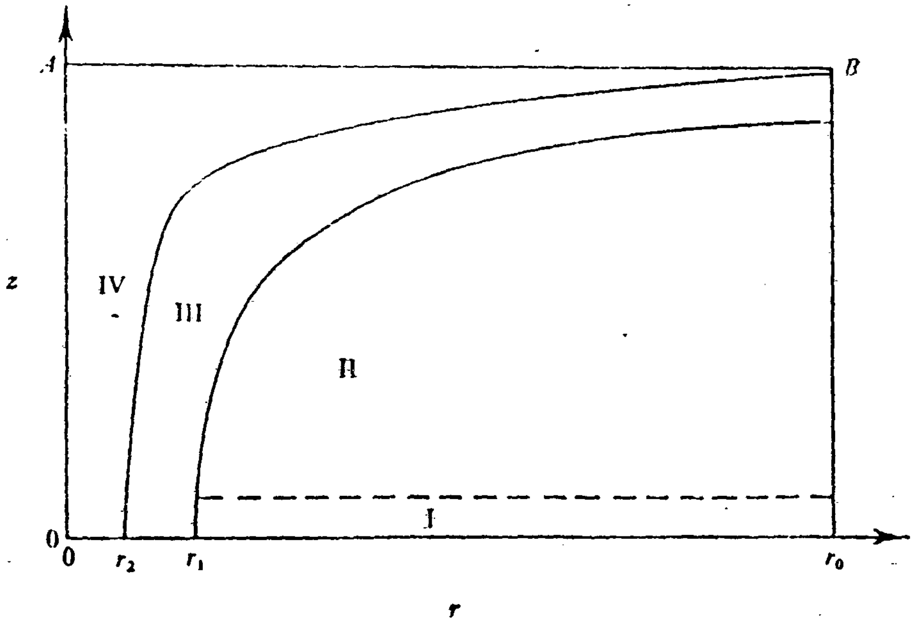

Figure 2 : Schematic Representation of a Hurricane.

Chapter 2

MATHEMATICAL FRAMEWORK FOR MODELLING OF TROPICAL CYCLONES

1. Introduction :

In this chapter we briefly review the basic equations from hydrodynamics and thermodynamics which govern the rotation of an air-parcel in the environment; these equations are then cast in a form useful for our purpose, viz, the description of tropical cyclones in the model of Charney and Eliassen.

2. The momentum Equations :

The fundamental law which governs motion of an air-percel is Newton's second law of motion :

$$\frac{d\vec{v}}{dt} = \sum_i \vec{F}_i \qquad (1)$$

where $\vec{v}$ is the velocity of the parcel of mass unity and the Summation is over the forces which act on it. Equation (1) holds in an inertial frame, whereas motion of the air-parcel takes place with respect to a rotating frame (which is non-inertial). As is well-known, we may use equation (1) for the description of motion of our air - parcel, provided we introduce the Coriolis and the Centrifugal forces in the summation. These are the so called "fictitious forces". The other forces (real forces) which must be included in the summation are, the pressure-gradient force, the force of gravity and the force of friction. Thus, equation (1) becomes

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\vec{\nabla}p + \vec{g} + \vec{F}_{f_r} \qquad \textbf{(3)}$$

Where $\vec{\Omega}$ is the angular velocity of the earth, ρ the density of the air parcel, $\vec{g}$ combines the gravitational and the centrifugal forces and $\vec{F}_{f_r}$ denotes the force of friction.

Equation (2) describes the motion of the air parcel moving w.r.t the rotating earth.

On the rotating earth, the unit reference vectors are clearly not constant, Therefore

$$\frac{d\vec{V}}{dt} = \vec{i}\frac{du}{dt} + \vec{j}\frac{dv}{dt} + \vec{k}\frac{dw}{dt} + u\frac{d\vec{i}}{dt} + V\frac{d\vec{j}}{dt} + w\frac{d\vec{k}}{dt} \qquad (3)$$

where (u, v, w) are the components of $\vec{V}$ and $\hat{i}, \hat{j}$ and $\hat{k}$ are the reference vectors. Terms like $\frac{d\vec{i}}{dt}$ may be calculated in terms of the latitude ϕ of the air parcel and the radius 'a' of the earth. Equation (3) then yields

$$\frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - \frac{uv}{a}\tan\phi + \frac{uw}{a}\right)\hat{\imath} + \left[\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{wv}{a}\right]\hat{\jmath}$$

$$+ \left[\frac{dw}{dt} - \frac{u^2+v^2}{a}\right]\hat{k} \qquad (4)$$

So that equation (2) may be separated into the equations :

$$\frac{du}{dt} = \frac{uv\tan\phi}{a} \frac{-uw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega\, v \sin\phi - 2\,\Omega\, w \cos\phi + F_x \qquad (5)$$

$$\frac{dv}{dt} = \frac{-u^2\tan\phi}{a} - \frac{vw}{a} - \frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega\, u \sin\phi + F_y \qquad (6)$$

$$\frac{dw}{dt} = \frac{u^2+v^2}{a} - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega\, u \cos\phi + F_z \qquad (7)$$

The "curvature" terms in the above equations are manifestly nonlinear ; so also is the operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla}$$

$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad (8)$$

These equations, therefore, Comprise an extremely complicated set of non-linear, coupled, partial differential equations. Furthermore, they describe all types and scales of atmospheric motions. The latter circumstance is a nuisance, when one is focussing attention on motion of a particular scale. To elaborate suppose we wish to describe the " Synoptical" scale motion in the atmosphere; we would then not be interested in using equations one solution of which is the sound waves. For this reason one resorts to so called scale analysis of the equations. This entails comparison of the various terms in the equations, keeping in mind the typical scales for the motion to be described. One then retains only the most dominant terms in the equations. This, of Course, also leads to significant mathematical simplicity. For mid-latitude synoptic scale motion in the absence of friction, it is easy to show that equations (5-7_ reduce to

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (9)$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (10)$$

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (11)$$

where f is the coriolis parameter

$$f = 2\,\Omega\,\mathrm{Sin}\,\phi \qquad (12)$$

3. **<u>The Conservation laws[3] :</u>**

A Conservation law is a consequence of the invariance of the equations of motion under a transformation. For example, the conservation of linear momentum follows from the translational invariance of the ~~energy~~ theory. In principle therefore, a conservation law gives no additional information, provided all the forces in a problem are known (which is the case in the meteorology) and if we are clever enough and have Computers of adequate speed and capacity to solve for the trajectories of all the particles of a system. Since 1 gm mole of air contains ~ 6 x 10^{23} molecules, it is clear that we must not attempt to solve a meteorological problem through the approach that relies on the knowledge of the trajectories of all the individual molecules that comprise the environment. Since, Conservation laws are independent of the details of the trajectories of the system and since they embody very general and significant consequences of the equations of motion, in practice, it proves to be of inestimable value to use these laws to obtain the solution of any given problem.

In the present study, we shall make use of three conservation laws, which are briefly dealt with below.

(i) **The law of Conservation of mass**

This law simply states that the mass inflow per unit volume in any region must equal the rate of mass increase per unit volume :

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} (\rho \vec{v}) = 0 \qquad (13)$$

Equation (13) is also referred to as the equation of Continuity.

(ii) **The law of Conservation of angular momentum**

At a latitude ϕ, if an air - parcel has relative (zonal) Velocity u, then its total (zonal) velocity is u + Ω cosϕ, where the second term is the velocity imparted to it by the rotation of the earth (v = Ω), consequently, the absolute (zonal) angular momentum m of the parcel (of mass unity) is given by

$$\begin{aligned}&(\text{angular momentum} = \vec{r} \times \vec{p})\\ &m = (u + \Omega\, a \cos\phi)\, a \cos\phi\end{aligned} \tag{14}$$

The law of conservation of angular momentum states that changes in m can come about only through the action of torque about the axis of rotation. The only such torques possible are due to east-west pressure gradients and friction. Therefore,

$$\begin{aligned}&\frac{dm}{dt} = -\left(\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x\right) a\mathrm{Cos}\phi\\ \text{or}\quad &\rho\frac{dm}{dt} = \left(-\frac{\partial p}{\partial x} + \rho F_x\right) a\mathrm{Cos}\phi\end{aligned} \tag{15}$$

Now

$$\rho\frac{dm}{dt} = \rho\frac{\partial m}{\partial t} + \rho\vec{V}\vec{\nabla}m$$

$$= \frac{\partial}{\partial t}(\rho m) - \frac{m\partial\rho}{\partial t} + \vec{\nabla}(\rho\, m\, \vec{v}) - m\vec{\nabla}.\ \rho\,\vec{v} \tag{16}$$

$$= \frac{\partial}{\partial t}(\rho m) + \vec{\nabla}(\rho\, m\, \vec{v}) - m\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}.\ \rho\vec{v}\right]$$

where the terms within the square brackets yield a null contribution because of equation (12). Hence equation (15) becomes

$$\frac{\partial}{\partial t}(\rho m) + \vec{\nabla}.\ (\rho_m\,\vec{v}) = \left(-\frac{\partial p}{\partial x} + \rho F_x\right) a\cos\phi \tag{17}$$

which is the equation for Conservation of angular momentum.

(iii) **<u>The potential temperature and the law of Conservation of Energy</u>**

Let a dry air-parcel have temperature T at pressure p. If its pressure is changed adiabatically to 1000 mb, its temperature would change to θ, which is called its potential temperature :

$$\theta = T\left(\frac{1000}{P}\right)^{R/C_p} \tag{18}$$

here R is the gas constant for dry air and C_p is the specific heat of air at constant pressure.

From equation (16) one may obtain

$$C_p \frac{d\,\ell n\theta}{dt} = C_p \frac{d}{dt}\,\ell nT - R\frac{d}{dt}\,\ell np \tag{19}$$

Recall now the First law of thermodynamics

$$\delta Q = C_v\, dT + pd\alpha \tag{20}$$

where dQ is the heat added to a system resulting in an increase in its internal energy (first term) and the performance of work (the second term). From equation (20)

$$\frac{dS}{dt} = C_p \frac{d\ell nT}{dt} - R\frac{d\ell np}{dt} \tag{21}$$

$\left(\delta S = \frac{\delta Q}{T}\right.$ is the change in entropy $\left.\right)$

($\delta S = \dfrac{\delta Q}{T}$ is the change in entropy}.

whence equations (19) and (21) yield

$$C_p \frac{d \ln\theta}{dt} = \frac{dS}{dt} \tag{22}$$

It is in this form that the First law of thermodynamics is used in meteorology. Note that for an adiabatic process ($\delta Q = 0$), the potential temperature is conserved.

(iv) **Tackling Friction**

As has been brought out in Chapter I, the typical tropical cyclone affects the entire troposphere including its lowest kilometer, the boundary layer. In this layer the Vertical Viscous force is generally of the same order of magnitude as the pressure gradient and coriolis forces. One must not therefore neglect the Viscous force, rather, as will be seen in the next chapter, the force of friction plays an important role in the development of the cyclones from a pre existing tropical depression. Nonetheless, a rigorous treatment of the force of friction to determine the structure of the velocity field in the vertical is not possible - Simply because it would require detailed knowledge of the structure and amplitude of the turbulent eddies which are responsible for the vertical momentum transport and the non - existent theory of turbulence.

(i) **The EKman layer**

In a classic work, Prandtl provided a theoretical basis for estimating the magnitude of the eddy viscosity. Since any problem in meteorology where frictional force is important makes use of this work - explicitly or implicitly, it is worthwhile to briefly review the essence of Prandtl's approach. To this end, we rewrite equations (9) and (10) - with the help of equations (8) and (12) - as

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) - f\rho v = -\frac{\partial p}{\partial x} \tag{23}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho vw) + f\rho v = -\frac{\partial p}{\partial y} \tag{24}$$

Prandtl's basic idea was that the momentum transport of small-scale eddy motions may be parameterized in terms of the large-scale mean flow. Thus

$$\begin{aligned} u &= \langle u \rangle + u' \\ v &= \langle v \rangle + v' \text{ etc.} \end{aligned} \tag{24}$$

where u is the instantaneous velocity that occurs in equations (22) and (23), $\langle u \rangle$ is the time-averaged velocity at any point and u' is the deviation from the mean (and thus associated with the turbulent eddy).

Substituting equation (24) into equations (22) and (23) and taking the time average of the latter equations over a time long enough to average out the fluctuations but short enough to preserve the trends in the large - scale flow. we obtain ---

$$\frac{\partial\langle u\rangle}{\partial t} + \langle u\rangle\frac{\partial\langle u\rangle}{\partial x} + \langle v\rangle\frac{\partial\langle u\rangle}{\partial y} + \langle w\rangle\frac{\partial\langle u\rangle}{\partial z} - f\langle v\rangle = \frac{-1}{\rho}\frac{\partial\langle p\rangle}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial z}\langle\rho u'w'\rangle \tag{25}$$

$$\frac{\partial\langle v\rangle}{\partial t} + \langle u\rangle\frac{\partial\langle v\rangle}{\partial x} + \langle v\rangle\frac{\partial\langle v\rangle}{\partial y} + \langle w\rangle\frac{\partial\langle v\rangle}{\partial z} + f\langle v\rangle = \frac{-1}{\rho}\frac{\partial\langle p\rangle}{\partial y} - \frac{1}{\rho}\frac{\partial}{\partial z}\langle\rho v'w'\rangle \tag{26}$$

where the horizontal gradient terms have been neglected since they are much smaller than the vertical gradient terms in the planetary boundary layer.

The eddy stress terms in equations (25) and (26) are usually parameterized as

$$-\rho\,\langle u'w'\rangle = A_z\frac{\partial\langle u\rangle}{\partial z} \tag{26 i}$$

$$-\rho\,\langle v'w'\rangle = A_z\frac{\partial\langle v\rangle}{\partial z} \tag{26 ii}$$

where A_z is the eddy exchange coefficient, assumed to be equal for x and y momentum. If one also assumes that the mean horizontal acceleration is negligible, equations (25) and (26) reduce to

$$-f\langle v\rangle = \frac{-1}{\rho}\frac{\partial\langle p\rangle}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial z}\left[A_z \frac{\partial\langle u\rangle}{\partial z}\right] \tag{27}$$

$$f\langle u\rangle = \frac{-1}{\rho}\frac{\partial\langle p\rangle}{\partial y} + \frac{1}{\rho}\frac{\partial}{\partial z}\left[A_z \frac{\partial\langle v\rangle}{\partial z}\right] \tag{28}$$

These equations become tractable under the further assumptions that ρ, A_z and the geostrophic wind are constant in the vertical. Then subject to the boundary conditions.

$u = 0, \ v = 0$, at $z = 0$

$u = ug, \ v = Vg$ at $z \to \infty$

One may show that equations (27) and (28) admit of the following solution ;

$$\left.\begin{aligned} u &= u_g\,(1 - e^{-\gamma z}\operatorname{Cos}\gamma z) \\ V &= V_g\, e^{-\gamma z}\operatorname{Sin}\gamma z \\ \text{where } \gamma &= \left(\frac{f}{2k}\right)^{1/2} \\ \text{and } k &= \left(\frac{A_z}{\rho}\right) \end{aligned}\right\} \tag{29}$$

Equation (29) represent the famous Ekman spiral. These equations determine the departure of the wind field from the geostrophic balance in the boundary layer. The height above the surface of the earth where the wind becomes geostrophic is referred to as the EKman layer; beyond this friction is usually negligible.

(v) The Charney - Eliassen Model

(i) Introduction

In the region of occurrence of tropical cyclones, the simultaneous occurrence of atmospheric motions of predominantly of two, vastly different scales is an important observational fact. These are the synoptic scale disturbances in which are embedded the cumulus motions. If the tropical atmosphere were conditionally unstable, its vertical thermal structure would be more favourable to small scale cumulus convection rather than to convective circulation of tropical cyclone scale. This led Charney and Eliassen (CE hereafter) to propose that the tropical atmosphere is governed by "Conditional instability of the second Kind" (CISK), which causes the pre-hurricane depression and cumulus cells to cooperate rather than compete for the same energy. Thus, while the cumulus cell drives the depression by supplying the heat energy, the depression supports the cumulus cell by producing the low-level convergence of moisture into the cell. The essence of the CE model is to show explicitly that such a coupling between the two scales of motion does lead to a large-scale self-amplification. The model concerns itself with the dynamics of the large scale motion the interaction of which with the cumulus-scale motion is suitably parameterized.

(ii) The basic equations

In the CE model the tropical cyclone is assumed to be represented by a balanced, axis-symmetric Vortex. This implies that pressure in the horizontal plane is only along the radial coordinate; also, it makes the use of cylindrical coordinates a natural choice. If u and v are respectively, the radial and the tangential components of horizontal velocity, equations (9) and (10) may be written as

$$-\rho\left[\frac{V^2}{r} + fv\right] - \frac{\partial p}{\partial r} = 0 \tag{32}$$

$$\frac{\partial v}{\partial t} + fu = 0 \tag{33}$$

We note that (a) the reduction of the prognostic equation (9) into the diagnostic equation (32) implies that time variations of the tangential circulation and mass fields are so slow that the two fields are always in a state of quasistatic equilibrium[4] ; (b) equation (33) is the perturbatively reduced form of equation (10).

We now deal with equation (12), the Continuity equation. CE argued that the local and horizontal fluctuations of pressure, density and temperature are small for the cyclone, enabling reduction of equation (12) in the cylindrical coordinates to

$$\frac{\partial}{\partial r}(\bar{\rho} r u) + \frac{\partial}{\partial z}(\bar{\rho} r w) = 0 \tag{34}$$

where a bar over a Variable denotes horizontal space - time average.

In Section III (ii), We ~~use~~ had dealt with the Conservation of angular momentum about the earth's axis of rotation. For the tropical cyclone, it is natural to consider the Conservation of angular momentum (m) about the axis of symmetry of the Cyclone viz,the local vertical. Since the projection of $\vec{\Omega}$ on to the local vertical of latitude ϕ is given by $\Omega \sin\phi$, we have for unit mass of air

$$m = rv + \frac{1}{2} fr^2 \tag{35}$$

with $\frac{\partial p}{\partial y} = 0$ and the neglect of friction, equation (15) may be written down in cylindrical coordinates as

$$\frac{\partial}{\partial t}(r\bar{\rho}m) + \frac{\partial}{\partial r}(r\bar{\rho}mu) + \frac{\partial}{\partial z}(r\bar{\rho}mw) = 0 \tag{36}$$

where m is given by equation (35)

It is convenient to combine the Constraint equations (34) and (36) into a single equation. For this purpose we write equation (36) as

$$r\bar{\rho}\frac{\partial m}{\partial t} + r\bar{\rho}u\frac{\partial m}{\partial r} + r\bar{\rho}w\frac{\partial m}{\partial z}$$

$$+ m\left[\frac{\partial}{\partial t}(r\bar{\rho}) + \frac{\partial}{\partial r}(r\bar{\rho}u) + \frac{\partial}{\partial z}(r\bar{\rho}w)\right] = 0 \tag{37}$$

The first term in the Square brackets in this equation is evidently Zero; The remaining two terms vanish by virtue of

equation (34). Multiplying each of the equations (36) and (37) by m and adding, we obtain

$$\frac{\partial}{\partial t}\left(\bar{\rho} r m^2\right) + \frac{\partial}{\partial r}\left(\bar{\rho} r u m^2\right) + \frac{\partial}{\partial z}\left(\bar{\rho} r w m^2\right) = 0 \qquad (38)$$

Using equation (8), we may write the equation of energy - conservation, equation (20), in cylindrical Coordinates as :

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial r} + w\frac{\partial \theta}{\partial z} = \frac{\bar{\theta}}{C_p}\frac{Q}{\bar{T}} \qquad (39)$$

Where Q is the rate of external heating per unit mass. Just as we combined the equations of Continuity and of angular momentum into one equation, we may combine the equations of Continuity and of energy - conservation into a single equation; in this manner the equation of continuity is dispensed with and the set of equations we have to deal with is reduced to one. To achieve this, we multiply equation (39) by ρ r and equation (37) by θ and add :

$$\frac{\partial}{\partial t}(\bar{\rho} r \theta) + \frac{\partial}{\partial r}(\bar{\rho} r u \theta) + \frac{\partial}{\partial z}(\bar{\rho} r w \theta) = \frac{\bar{\rho} r \theta}{C_p T} Q \qquad (40)$$

Equation (40) brings out how the heating provided by the cumulus cell drives the large-scale motion of the cyclone ; the manner in which moisture is converged into the cumulus cell by the large- scale motion will be dealt with later.

Since the coriolis parameter is treated as a constant in the CE model, angular momentum (m) is a function of the radial coordinate (r) and the tangential velocity (v) only. This enables us to write the momentum equation (32) in terms of m. we have

$$
\begin{aligned}
-\rho\left(\frac{v^2}{r} + fv\right) &= -\frac{\rho}{r^3}(v^2r^2 + fvr^3) \\
&= \frac{-\rho}{r^3}\left[\left(vr + \frac{1}{2}fr^2\right)^2 - \frac{1}{4}f^2r^4\right] \\
&= \frac{-\rho}{r^3}\left[m^2 - \frac{1}{4}f^2r^4\right]
\end{aligned}
$$

Whence equation (32) becomes

$$-\rho \left[\frac{m^2}{r^3} - \frac{1}{4}f^2r\right] = \frac{\partial p}{\partial r}$$

$$\text{or } \frac{m^2}{r^3} = \frac{-1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{4} f^2r = \frac{\partial \chi}{\partial r} \qquad (41)$$

$$\text{where } X = \frac{p-\bar{p}}{\bar{\rho}} + \frac{1}{8} f^2 r^2 \qquad (42)$$

It should be noted that the assumption that the vortex is axis-symmetry implies that the condition of balance in the radial direction holds at each horizontal level as we move along the local vertical. The ρ in equation (42) is representative of the density of the horizontal level at which (42) holds, and therefore a constant.

(iii) **Incorporation of friction and the heating by Condensation**

Equation (41), (10), (11), (38), and (40) are the basic equations in the CE model. These equations - or variants there of - have also been of central importance in the scores of attempts to model the tropical cyclone after the work of CE. However the CE model is as yet incomplete in two respects:

(a) The Force of friction has not been taken into account and (b) the mode of estimating Q, the rate of external heating provided by the cumulus cell, has not been dealt with. It is in the treatment of these aspects that various models differ. For reasons spelt out in next chapter, the incorporation of friction must involve some sort of parameterizaiton. Because in general the cumulus cell has a scale two orders of magnitude smaller than the cyclonic scale, it is convenient also to estimate Q through parameterization, rather than an independent set of equations. These parameterizations require ingenuity on the part of the model builder. An account of the manner in which CE dealt with them is given in the next chapter

REFERENCES

1. J.G. Charney and A. Eliassen, J. Atmos. Sci. 21, 68-75 1964

2. See for example, J.R. Holton, An Introduction to Dynamic Meteorology (Academic, N.Y.) 1972.

   G.J. Haltiner and F.L. Martinet, Dynamic and Physical Meteorology (McGraw Hill, N.Y.) 1957.

   S.L. Hess, Introduction to Theoretical Meteorology (Holt, RineHart and Winston, N.Y.) 1959.

3. C. Kittel and W.D. Knight, Mechanics (Tata McGraw Hill, Bombay) 1965.

Chapter 3

THE CHARNEY - ELIASSEN MODEL

I. Summary of basic equations

The Charney-Eliassen (CE) model was introduced in the previous chapter. Its basic premise is that the tropical cyclone develops by a conditional instability of the second kind (CISK) in which the cumulus and cyclone-scale motions grow together rather than one of them growing at the expense of the other.

Under the assumptions that (a) the cyclone may be represented by an axis-symmetric vortex (b) The coriolis parameter is constant over the region of formation of the cyclone and (c) the fluctuations of pressure, density and temperature are small in a horizontal plane as we move along the local vertical, the following basic equations in the model were derived in the last chapter.

The horizontal momentum equations :

$$\frac{\partial \chi}{\partial r} = \frac{m^2}{r^3} \tag{1}$$

$$\frac{\partial v}{\partial t} + fu = 0 \tag{2}$$

The vertical momentum equation (hydrostatic approximation equation):

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = g \qquad (3)$$

The Combined equations of continuity and of angular momentum Conservation :

$$\frac{\partial}{\partial t}(\overline{\rho}\ r\ m^2) + \frac{\partial}{\partial r}(\overline{\rho} r\ u\ m^2) + \frac{\partial}{\partial z}(\overline{\rho} r\ w\ m^2) = 0 \qquad (4)$$

The Combined equations of Continuity and energy Conservation :

$$\frac{\partial}{\partial t}(\overline{\rho}\ r\ \theta) + \frac{\partial}{\partial r}(\overline{\rho} r\ u\ \theta) + \frac{\partial}{\partial z}(\overline{\rho} r\ w\ \theta) = \frac{\overline{\rho}\ r\ \theta\ Q}{C_p\ T} \qquad (5)$$

To complete the description of the model we need to specify the parameterizations of friction and of cumulus heating. Before we deal with these, let us detail the vertical structure of the tropical atmosphere used by CE.

II. Vertical Structure of the topical atmosphere in the model

The CE model is a 2-level model, at each of which equations (1-5) hold with appropriate specification of variables. The levels to which the model equations are applied are numbered 1 and 3; variables at these levels are estimated by the method of finite-differencing applied to the variable-values at levels 0,2 and 4. Level 4 marks the end of the friction layer. These details of stratification used by CE are given in Figure 1.

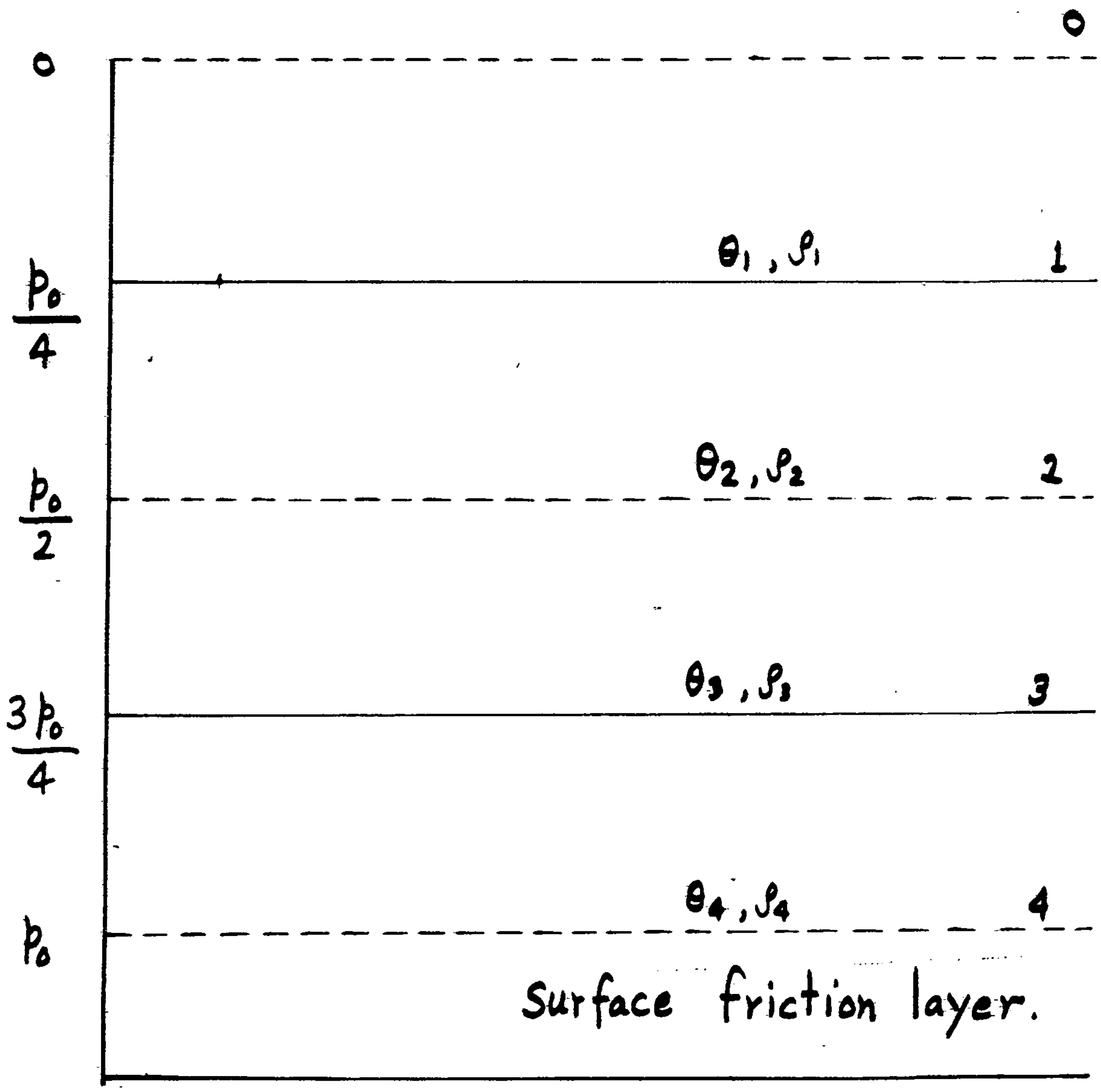

Figure 1: 2-layer model of a tropical cyclone

III. **The role of friction :**

One normally associates friction with the dissipation of energy. One would imagine, therefore, that friction ought to be the cause for surface winds in the tropics to lose energy. CE argued, however, that in the tropics surface friction also caused evaporation from the oceans which subsequently results in the release of latent heat. Overall, it plays the role of an energy-creating mechanism. To incorporate the effect of friction in their model CE noted that in the perturbation analysis, the equation for balance of forces in the radial direction, equation (2.32) reduces to

$$-\rho f v - \frac{\partial p}{\partial r} = 0 \qquad (6)$$

i.e. to an equation for quasi-geostrophic motion. For such motions, Charney and Eliassen had shown earlier that the frictionally induced vertical velocity at the top of Ekman layer is given by

$$\omega_o = \frac{1}{2} D_E \, \zeta_g \, \mathrm{Sin} 2\alpha \qquad (7)$$

Where $D_E = \left(\frac{2A}{f}\right)^{1/2}$ is a measure of the depth of the Ekman layer, A is the (Constant) Kinematic eddy Coefficient of Viscosity, α is the angle between the surface geostrophic wind and the surface isobars, and ζ_g is the vorticity of the surface geostrophic wind.

Since the top of the Ekman layer in the CE model has been designated as level 4, and since

$$\zeta = \hat{e}_z \nabla \times \vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_4)$$

in the axis-symmetric model, the entire effect of friction is to provide the following relation between the vertical and horizontal components of wind at the said level.

$$\omega_4 = \frac{D_E \operatorname{Sin} 2\alpha}{2r} \frac{\partial}{\partial r}(rv_4)$$

IV. **The Stream functions**

The continuity equation discussed in the previous chapter, viz;

$$\frac{\partial}{\partial r}(\bar{\rho} r u) + \frac{\partial}{\partial z}(\rho \bar{r} \omega) = 0$$

enables one to write

$$\bar{\rho} r u = \frac{-\partial \Psi}{\partial z}, \bar{\rho} r \omega = \frac{\partial \Psi}{\partial r} \qquad (9)$$

Where ψ is the stream function. The analysis in the CE model is facilitated by the use of this function. Equations (9) and (3) imply that

$$\begin{aligned} ru_1 &= \frac{g}{\Delta p}\psi_2, \quad r\bar{\rho}_2\omega_2 = \frac{\partial \psi_2}{\partial r} \\ ru_3 &= \frac{g}{\Delta p}(\psi_4 - \psi 2), \quad r\bar{\rho}_4\omega_4 = \frac{\partial \psi_4}{\partial r} \end{aligned} \qquad (10)$$

where

$$\Delta\bar{p} = \bar{p}o/2$$

V. **Incorporating friction in one of the horizontal momentum equations**

The CE model considers the tropical depression to be a small perturbation of a state of relative rest. The objective then is to investigate how the coupling between the cumulus and the tropical-scale affects the time evolution of the perturbation.

Using the appropriate equation from the set of equations (10), the perturbed form of equation (8) is:

$$\frac{1}{\bar{\rho}_4 r} \frac{\partial \psi^{1_4}}{\partial r} = \frac{D_E \, Sin2\alpha}{2r} \frac{\partial}{\partial r}(rv_4^1)$$

Integration w.r.t r and differentiation w.r.t t gives

$$\frac{\partial \psi_4^1}{\partial t} = \frac{\bar{\rho}_4 D_E \; Sin2\alpha}{2} r \frac{\partial}{\partial t}(V_4^1) \tag{11}$$

Equation (11) represents the role that friction plays in the problem. This equation may be combined with the horizontal momentum equation (2); the perturbed form of which at level 4 is

$$\frac{\partial v_4^1}{\partial t} + fu_4^1 = 0 \tag{12}$$

From the first of equations (9) and equation (3), we have

$$u'_4 = \frac{g}{r}\left(\frac{\partial\psi}{\partial p}\right)_4 \qquad (13)$$

$\left(\frac{\partial\psi}{\partial \rho}\right)_4$ may be estimated by making a Taylor expansion about

the level 2 :

$$\psi'_2 = \psi'_4 + (\bar{p}_2-\bar{p}_4)\left(\frac{\partial\psi_4}{\partial p}\right)_4$$

$$whence \left(\frac{\partial\psi_4}{\partial p}\right)_4 = \frac{\psi'_2-\psi'_4}{-\left(\frac{po}{2}\right)} \qquad (14)$$

Substituting equations (11), (13) and (14) into the equation (12), we obtain

$$\frac{\partial\psi'_4}{\partial t} = K(\psi'_2-\psi'_4) \qquad (15)$$

$$\text{where } K = \left(\frac{D_E}{H}\right) f\ Sin2\alpha \qquad (16)$$

$$H = \frac{RT_4}{g} \qquad (17)$$

and we have used the gas equation. Equation (15) is a horizontal momentum equation which incorporates the effect of friction.

VI. **Parameterization of heating by Condensation:**

The CE model, at this stage, remains incomplete only to the extent that we have not dealt with the estimation of the external heating rate, Q, in equation (5). As stressed by CE, this is the most difficult task in the model. Strictly speaking, the determination of Q requires a system of equations to describe the turbulent transport properties of the cumulus convection field in statistical equilibrium with the large-scale field of motion. Since a self consistent theory for such a system does not exist, one is forced to parameterize the process - just as one did for the incorporation of friction.

We recall from section V (i) of chapter 2 that the cyclone develops in a region in which the cumulus cells are embedded in the large-scale depression. CE considered a vertical cylindrical column of air in the tropical atmosphere, extending to the top of the troposphere, whose horizontal cross-section is large enough to contain several cumulus cells and yet small enough to be regarded as infinitesimal w.r.t the large scale depression. CE assumed that radial flux of moisture is given by $\bar{\rho}\,\bar{u}\,\bar{q}$ where $\bar{q}$ is the specific humidity (bar denotes horizontal average). Then horizontal convergence of moisture into a unit vertical column is given by

$$= \int_0^\infty - \vec{\nabla} (\bar{\rho}\ \bar{u}\ \bar{q}) dz$$

$$= \int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} (r\ \bar{\rho}\ \bar{u}\ \bar{q}) dz$$

Further, for lack of better knowledge $\bar{q}$ is assumed to be a known fraction μ (z) of the mean saturation value $\bar{q}_s$ (r,z,t). Thus the total convergence of moisture into a vertical unit column becomes

$$I = -\int_0^\infty \frac{\mu}{r} \frac{\partial}{\partial r} (\bar{\rho} r\ \bar{u}\ \bar{q}_s) + (\bar{\mu}\ \bar{\rho}\ \bar{w}\ \bar{q}_s)_g \tag{18}$$

where g denotes top of the friction layer.

If the small radial variation of q_s is ignored and the integral in equation (18) evaluated by parts after putting

$$\frac{\partial}{\partial r} (r\ \bar{\rho}\ \bar{u}) = -\frac{\partial}{\partial z} (r\ \bar{\rho}\ \bar{w})$$

(by virtue of the equation of continuity)

We obtain

$$I = -\int_0^\infty \rho \frac{\partial}{\partial z} (\mu \bar{q}_s) dz \tag{19}$$

If the liquid water content of the air is small, the corresponding release of latent heat in the column is L I (L is the mean latent heat of condensation), which is assumed to be distributed in the vertical in proportion to the heat released from a parcel of saturated air ascending moist adiabatically. Thus,

$$\bar{\rho}\, Q = \frac{-L}{q_{sg}} \frac{d\bar{q}_s}{dz} I \tag{20}$$

inside the cumulus cell, outside of this zone Q = 0

From equation (19)

$$I = -\int_0^{\infty} \bar{\rho} w \frac{\partial}{\partial z}(r\bar{\rho}\, \bar{w})$$

$$= -\int_0^{\bar{p}0} \bar{\rho}\, \bar{w} \frac{\partial}{\partial p}(\mu \bar{q}_s)\, dp$$

$$= \mu \frac{\partial \bar{q}_s}{\partial p} \Delta p [\bar{\rho}_2 w_2 + \frac{1}{2}\bar{\rho}_4 w_4]$$

$$= \mu \frac{\partial \bar{q}_s}{\partial p} \Delta p [\frac{1}{r}\frac{\partial \psi_2}{\partial r} + \frac{1}{2r}\frac{\partial \psi_4}{\partial r}]$$

Where we have used the hydrostatic approximation, the extended trapezoidal rule of integration with reference to levels 0, 2 and 4, and the definitions given in equations (10). Hence equation (20) gives

$$\begin{aligned} Q_1 = Q_3 &= \frac{g\mu L}{q_s g} \frac{\partial \bar{q}_s}{\partial p} \frac{\partial \bar{q}_s}{\partial p} \Delta p \frac{1}{r}\frac{\partial}{\partial r}(\psi_2 + \frac{1}{2}\psi_4) \\ &\simeq \frac{g\mu L}{2\Delta p}(\bar{q}_{s3} - \bar{q}_{si}) \frac{1}{r}\frac{\partial}{\partial r}(\psi_2 + \frac{1}{2}\psi_4) \end{aligned} \tag{21}$$

VII. <u>Incorporation of Conservation of angular momentum and of energy in the second horizontal momentum equation</u>

From the definition of potential temperature given in equation (16) of chapter 2, we have

$$\ln \theta = \frac{1}{\gamma}\ln p - \ln \rho + \text{Const}$$

$$\gamma = \frac{C_p}{C_v}$$

We also recall here the definition of χ

$$\chi = \frac{p-\overline{p}_0}{\rho} + \frac{1}{8}f^2r^2$$

and the equation of hydrostatic approximation

$$\frac{-1}{\rho}\ \frac{\partial p}{\partial z} = g$$

If we assume that $D\ \frac{\partial \ln \overline{\theta}}{\partial z}$, representing the fractional change of θ in the depth D, is small we find that

$$\begin{aligned} \ln\theta &= \ln\overline{\theta} + \frac{1}{g}\frac{\partial \chi}{\partial z} \\ &= \ln\overline{\theta} - \rho\frac{\partial \chi}{\partial p} \end{aligned} \tag{23}$$

We now consider the remaining horizontal momentum equation, equation (1), at level 2:

$$\frac{\partial \chi_2}{\partial r} = \frac{m_2^2}{r^3}$$

Differentiation w.r.t $\bar{p}$ and t gives

$$\frac{-\partial}{\rho_2\theta_2}\frac{\partial^2\theta_2}{\partial r\partial t} = \frac{1}{r^3}\frac{\partial^2 m^2}{\partial p\partial t} \tag{24}$$
$$\approx \frac{1}{r^3\Delta p}\frac{\partial}{\partial t}(m_3^2 - m_1^2)$$

Where we have used equation (23). The strategy now is to combine equation (24) with the constraint equations for angular momentum and energy, viz. equations (4) and (5). For levels 1 and 3, these equations give

$$\frac{\partial}{\partial t}(rm_1^2) + \frac{\partial}{\partial r}(ru_1 m_1^2) + \frac{g}{\Delta p}(-\bar{\rho}_2 r w_2 m_2^2) = 0$$
$$\frac{\partial}{\partial t}(rm_3^2) + \frac{\partial}{\partial r}(ru_3 m_3^2) + \frac{g}{\Delta p}(-\bar{\rho}_2 r w_2 m_2^2) = 0 \tag{25}$$

$$\frac{\partial}{\partial t}(r\theta_1) + \frac{\partial}{\partial r}(ru_1\theta_1) + \frac{g}{\Delta p}(-\bar{\rho}_2 r w_2\theta_2) = \frac{r\bar{\theta}_2 Q}{C_p T_1}$$
$$\frac{\partial}{\partial t}(r\theta_3) + \frac{\partial}{\partial r}(ru_3\theta_3) + \frac{g}{\Delta p}(\bar{\rho}_2 r w_2\theta_2 - \bar{\rho}_4 r w_4\theta_4) + \frac{r\bar{\theta}_3 Q_3}{C_p T_3} \tag{26}$$

In equation (25) and (26), we may approximate $m_2{}^2$ by $\frac{(m_1^2 + m_3^2)}{2}$ and θ_2 by $\frac{\theta_1 + \theta_3}{2}$. Further we replace

ru_1 by $(\frac{-g}{\Delta p})\psi_1, \bar{\rho}_2 r w_2$ by $\frac{\partial\psi_2}{\partial r}$, etc. (See equation (10)), to obtain

$$\frac{\partial}{\partial t}(m_3^2-m_1^2) = \frac{1}{r}\frac{g}{\Delta p}\left[\psi_2\frac{\partial}{\partial r}(m_1^2+m_3^2) - \frac{\partial}{\partial r}(m_3^2\psi_4) + m_4^2\frac{\partial\psi_4}{\partial r}\right] \tag{27}$$

$$\frac{\partial\theta_2}{\partial t} = \left(\frac{-g}{2\Delta p}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\{\psi_2(\theta_1-\theta_3)+\theta_3\psi_4\}\frac{-\theta_4}{r}\frac{\partial\psi_4}{\partial r}\right] + \frac{1}{2}\left(\frac{\bar{\theta}_1 Q_1}{C_pT_1} + \frac{\bar{\theta}_3}{C_pT_3}Q_3\right) \tag{28}$$

Substitution of these equations into equation (24) gives

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[(\theta_1-\theta_2)\psi_2+\theta_3\psi_4\right] - \frac{\theta_4}{r}\frac{\partial\psi_4}{\partial r}\right\}$$
$$\frac{-2\bar{\theta}_2}{\rho\bar{T}_2 r^4}\left\{\psi_2\frac{\partial}{\partial r}(m_1^2+m_3^2) - \frac{\partial}{\partial r}(m_3^2\psi_4) + m_4^2\frac{\partial\psi_4}{\partial r}\right\}$$
$$= \frac{\Delta p}{g}\frac{\partial}{\partial r}\left(\frac{\bar{\theta}_1}{C_pT_1}Q_1 + \frac{\bar{\theta}_3}{C_pT_3}Q_3\right) \tag{29}$$
$$= \frac{\mu L}{2}\left(\frac{\bar{\theta}_1}{C_pT_1} + \frac{\bar{\theta}_3}{CpT3}\right)(\bar{q}_{s3}-\bar{q}_{s1})\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(\psi_2 + \frac{1}{2}\psi_4)\right]$$

Where $\bar{p}_2 = \bar{\rho}_2 RT_2$,

and we have used equation (21).

Just as we considered the either momentum equation in its perturbed form, so also we now regard the tropical cyclone to be a small perturbation of a state of relative rest. This implies that

and

$$m_1 = m_3 - \frac{1}{2} f r^2$$

$$\text{so that } \frac{\partial m_1}{\partial r} = \frac{\partial m_3}{\partial r} = fr \tag{30}$$

$$\text{and } \frac{\partial m_1^2}{\partial r} + \frac{\partial m_3^2}{\partial r} = f^2 r^3$$

$$\text{Defining } \lambda^2 = \frac{4f^2}{RT_2} \frac{\overline{\theta}_2}{\theta_3 - \theta_1}$$

$$\kappa = \frac{\frac{L}{2}\left(\frac{\overline{\theta}_1}{C_p T_1} + \frac{\overline{\theta}_3}{C_p T_3}\right)(\overline{q}_{s3} - \overline{q}_{s1})}{\overline{\theta}_1 - \theta_3} \tag{31}$$

and letting primes denote perturbations, we may write equation (29) as:

$$r \frac{\partial}{\partial r}\left\{\frac{1}{r}\left[(1-\kappa\mu)\frac{\partial \psi_2'}{\partial r} - \frac{\mu\kappa}{2}\frac{\partial \psi_4'}{\partial r}\right]\right\} - \lambda^2\left(\psi_2' - \frac{1}{2}\psi_4'\right) = 0 \tag{32}$$

Equation (32) is the equation we sought , i.e., the equation which combines equations (1), (4) and (5).

VIII. The reduction of the model equations to an eigen value equation

We note that the entire content of the model is now contained in equations (15) and (32) which are reproduced below for covenience:

$$\frac{\partial \psi_4'}{\partial t} = K\ (\psi_2' - \psi_4') \tag{15}$$

$$r \frac{\partial}{\partial r}\left\{\frac{1}{r}\left[(1-\kappa\mu)\frac{\partial\psi_2'}{dr} - \frac{\kappa\mu}{2}\frac{\partial\psi_4'}{\partial r}\right]\right\} - \lambda^2\left(\psi_2' - \frac{\psi_4'}{2}\right) = 0 \qquad (32)$$

where K, λ and κ are given by equations (16), (30) and (31) respectively. Seemingly κ is rather complicated function of many variables, it is shown in appendix A that, in fact, it may be related to simply the derivative of potential equivalent temperature w.r.t. the potential temperature, viz,

$$\kappa \simeq \left(1 - \frac{\ln \overline{\theta}_E}{\ln \theta}\right)$$

In order to determine the time evolution of the system, we let.

$$\psi_2' = \overline{\Psi}_2\, e^{\sigma t}\ , \quad \psi_4' = \overline{\Psi}_4\, e^{\sigma t} \qquad (34)$$

substituting equations (34) into equations (15) and (32), and eliminating ψ_4, we get

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d\psi_2}{dr}\right) - \left(\frac{\sigma/k + 1/2}{\sigma/k + 1}\right)\frac{\lambda^2\ \psi_2}{\left[1 - \kappa\mu\left(\frac{\sigma/k + 3/2}{\sigma/k + 1}\right)\right]} = 0 \qquad (35)$$

The above equation holds both for the region of ascending motion and the region of descending motion. Since descending air is essentially dry, its potential equivalent temperature equals its potential temperature, Hence for descending motion, equation (35) becomes :

$$r \frac{d}{dr}\left(\frac{1}{r} \frac{d\bar{\Psi}_-}{dr}\right) - \lambda^2_- \bar{\Psi}_- = 0 \tag{36}$$

where

$$\lambda^2_- = \left(\frac{\sigma/k + 1/2}{\sigma/k + 1}\right) \lambda^2 \tag{37}$$

and ψ_- describes descending air. For ascending (moist) air, since $\kappa > 1$, equation (35) may be written as where ψ_+ describes ascending air, and

$$r \frac{d}{dr}\left(\frac{1}{r} \frac{d\bar{\Psi}_4}{dr}\right) + \lambda^2_+ \bar{\Psi}_+ = 0 \tag{38}$$

$$\lambda^2_+ = \left[- 1 + \kappa\mu\left(\frac{\sigma/k + 3/2}{\sigma/k + 1}\right)\right]^{-1} \lambda^2_- \tag{39}$$

with the change of variables

$$\bar{\Psi}_- = r\,\phi\ ,\ r = R/\lambda_-$$

equation (36) becomes

$$R^2\,\phi''(R) + R\,\phi'(R) - (R^2 + 1)\,\phi(R) = 0$$

which is recogonised as the modified Bessel equation, whose general solution involves K_1 and $I_{\pm 1}$. The boundary condition $r\ u \to 0$ as $r \to \infty$ then dictates that we have

$$\bar{\Psi}_-(r) = A_-\, r\, K_1(\lambda_-\, r) \tag{40}$$

Similarly, the acceptable solution of equation (38) is

$$\psi_+(r) = A_+ r J_1(\lambda_+ r) \tag{41}$$

At $r = a$, the radius separating the rising and sinking motions, the continuity of ψ implies that

$$\begin{aligned} &\overline{\Psi}_+(a) = \overline{\Psi}_-(a), \\ &\text{or } A_+ J_1(\lambda_+ a) = A_- K_1(\lambda_- a) \end{aligned} \tag{42}$$

From equation (36), we obtain

$$\begin{aligned} &\frac{1}{\lambda^2_-} \int_a^\infty \frac{d}{dr}\left(\frac{1}{r}\frac{d\overline{\Psi}_-}{dr}\right) dr = -\int_a^\infty \frac{\psi_-}{r}\, dr \\ &\text{or} \qquad \frac{1}{\lambda^2_-}\frac{d\,\overline{\Psi}_-(a)}{dr} = a\int_0^a \frac{\psi_-(r)}{r}\, dr \end{aligned} \tag{43}$$

Since $\psi_-(\infty) = 0$:

Similarly from equation (38) we obtained

$$\frac{1}{\lambda^2_+}\frac{d\overline{\Psi}_+(a)}{dr} = -a\int_0^a \frac{\psi_+(r)}{r}\, dr \tag{44}$$

since $\psi_+(0) = 0$

Upon adding equations (43) and (44) we obtain

$$\frac{1}{\lambda^2_-}\frac{d\overline{\Psi}_-(a)}{dr} + \frac{1}{\lambda^2_+}\frac{d\overline{\Psi}_+(a)}{dr} \tag{45}$$

Since the mass of ascending air must equal the mass of descending air

From equation (43) and (44), respectively, one may easily show that.

$$\frac{d\,\overline{\Psi}_-(a)}{dr} = -\,a\,\lambda_-\,A_+\,k_0\,(\lambda_- a)$$

and

$$\frac{d\,\overline{\Psi}_+(a)}{dr} = a\,\lambda_+\,A_+\,J_0\,(\lambda_+ a)$$

so that equation (45) becomes

$$\frac{A_+}{\lambda_+}\,J_0\,(\lambda_+ a) = \frac{A_-}{\lambda_-}\,K_0\,(\lambda_- a) \tag{46}$$

Finally, equations (42), and (46) may be combined to yield

$$\frac{J_1\,(\lambda_+ a)}{J_0\,(\lambda_+ a)} = \left(\frac{\lambda_-}{\lambda_+}\right)\frac{K_1\,(\lambda_- a)}{K_0\,(\lambda_- a)} \tag{47}$$

Equation (47) relates "solution" in the region ~~as~~ of rising motion to the "solution" in the region of sinking motion, and is hence of the form of an eigenvalue equation; specifically, equation (47) expresses the relation that must hold between σ/k and a and $\kappa\ \mu$ in order that the model equations (15) and (32) have a consistent solution. We note finally that equation (47) was obtained by CE in a slightly different form, which involved Hankel functions of imaginary argument. We have preferred to use the modified Bessel functions in order to make the reality of the equation manifest.

IX. **Numerical solutions**

Equation (47) is solved numerically. To this end, we first need to fix μ and κ. From the conditions of the tropical atmosphere in the hurricane season, one may estimate κ form equation (33) as 1.1. We now choose different values of μ in the range (0.7, 1) and, corresponding to each of these, seek to determine the set (Σ) of values of λa and σ/k which satisfies the equation.

To determine Σ, we choose a value for $\lambda_- a$ and guess a value for σ/k. This enables us to calculate $\lambda_+ a$ (see equation 39), Since κ, μ, σ/k and $\lambda_- a$ are now known.
The values of $k_1(\lambda_- a)$ and $k_0(\lambda_- a)$, for $\lambda_- a$= 0.1, 0.2, ..etc., are then read off from the tables of these functions given in Abramowitz and Stegun[2]; for $\lambda_- a$ < 0.1; we use the approximate formulae :

$$K_0\ (z) \sim -\ln z$$
$$K_1\ (z) \sim \frac{1}{2}\left(\frac{1z}{2}\right)^{-1} \qquad (48)$$

There now remains the problem of determining the ratio J_1/J_0. at any arbitrary value of the arguments of these functions; for this purpose we use the infinite continued fraction representation.

In this manner the function F_j,

$$\frac{J_\nu(z)}{J_{\nu-1}(z)} = \frac{1}{2\nu z-1} \; \frac{1}{-\;2(\nu+1)z^{-1}} \; \frac{1}{-\;2(\nu+2)z^{-1}} \tag{49}$$

$$(z>1)$$

$$= \left[\frac{1/2z/\nu}{1\;-}\;\frac{1/4z^2/[\nu\;(\nu+1)]}{1\;-}\;\frac{1/4\;z^2/[(\nu+1)\;(\nu+2)]}{1\;-}\qquad\right]$$

$$F_j = s\,\frac{J_1(\lambda_+ a)}{J_0(\lambda_+ a)} - \frac{K_1(\lambda_- a)}{K_0(\lambda_- a)}\;,\; s = \left(\frac{\lambda_+}{\lambda_-}\right) \tag{50}$$

can be determined; by varying $\lambda_- a$, and finding the corresponding value of σ/k for which this function vanishes, we can determine the set Σ. The zeros of the function are found by the method of chords. The computer programme used for numerical work is given in Appendix B.

The results obtained for different values of μ are given in Fig 3. Before we put the significant findings of the CE model in focus, let us note that for $f = 2\Omega \sin\phi = 0.377 \times 10^{-4}$ (for $\phi = 15°$), $p_0 = 1000$ mb and $H = 8.0$ km, we have $D_E = \sqrt{2A/f} =$ 0:73 km (for $A = 10 m^2\ sec^{-1}$), and $k = \mathrm{sing}\ 2\ \alpha\ (D_E/H) = 1.72 \times 10^{-6}\ sec^{-1}$ (for $\alpha = 15°$)

These values have to be used to determine the scales of a and σ after $\lambda_- a$ and σ/k are a determined as described above.

We end this chapter by enumerating below the significant findings of the CE model:

(a) From the relation

$$\psi_2'(r,t) = \Psi_2(r)\ e^{\sigma t}$$

and the fact that σ turns out to be positive, it follows and the fact that σ turns out to be positive, it follows that the pertubation of the pre-hurricane depression does grow in the model. For $t = \sigma^{-1}$, we find that ψ'_2 ($t = \sigma^{-1}$) $= e\ \psi'_2$ ($t = 0$). Thus, σ^{-1} signifies the time after which the perturbation grows e-fold.

(b) The plot of σ/k vs λ_a (Fig. 3) is equivalent to a plot of σ vs. 'a'. We recall that 'a' is the value of the radius from the edge of the hurricane which separates the regions of rising and sinking motions. Thus, 'a' signifies the region of convection. From Fig.3 we find that for realistic values of μ for the tropical atmosphere in the hurricane season, i.e. for $0.7 \le \mu \le 0.8$, σ lies in the range 10^{-6} to 10^{-5} sec^{-1}, giving an e-folding time in the range 10 days to 1 day.

(c) For $\mu = 0.8$, $\sigma^{-1} \simeq$ 2-5 days and a $\simeq$ 100 km, beyond which σ falls off monontonically. Thus, a $\simeq$ 100 km is the order of magnitude of the size of the active convective region of the unstable disturbance.

(d) We note that the model does not concern itself with the prediction of tangential velocities as a function of r, the amount of rainfall as a function of time after the initial depression is assumed to exist, rather, its focus is on the idea that a depression can grow spontaneously under the conditions spelled out in the model.

REFERENCES

1. J.G. Charney and A. Eliassen, J. Atmos. Sci, 21, 68-75, 1964.

2. M. Abramowitz and I.A. Stegum, Handbook of Mathematical Functions, Dover Publications, NY, 358-417, 1972.

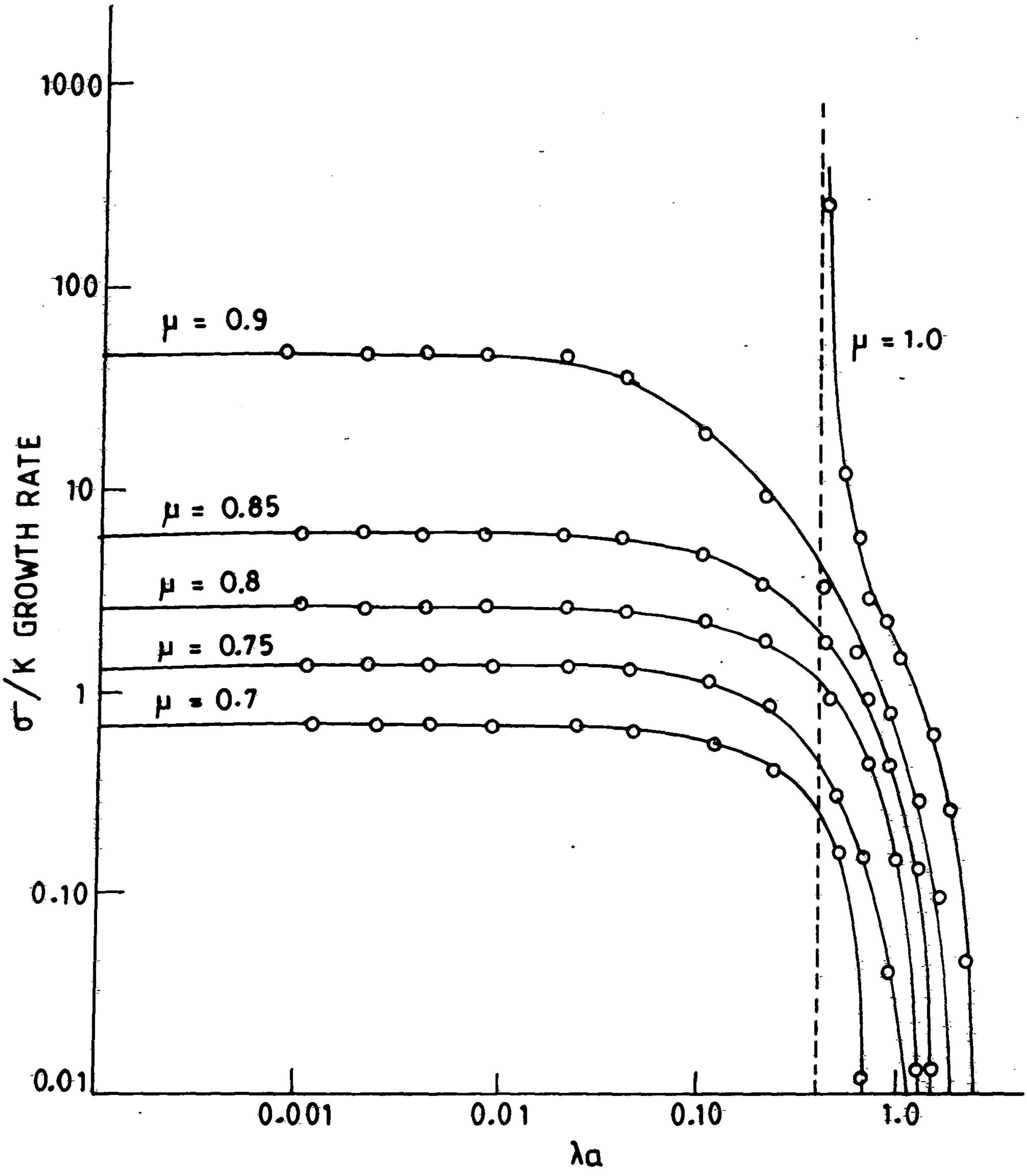
σ/K GROWTH RATE
1000
100
10
1
0.10
0.01
0.001
0.01
0.10
1.0
λa
μ = 0.9
μ = 0.85
μ = 0.8
μ = 0.75
μ = 0.7
μ = 1.0

Figure :3

Chapter 4

OVERVIEW OF VARIOUS MODELS.

The genesis, intensification and maintenance of tropical cyclones have been studied widely during the past three decades because of their spectacular nature in the atmosphere. Many observational, theoretical, numerical and dynamical studies have been carried out. In this chapter we are providing the gist of some of such important models and their important characteristics.

K. Ooyama proposed a dynamical model in 1963. Like charney - Eliassen model (discussed in earlier chapter) this model also treats cyclone generation as a cooperative process between the cloud-scale convection and the cyclone scale motion. In order to formulate the coupling between the two different scales of phenomena, it was hypothesized that the statistical distribution and mean intensity of the cloud convection are controlled by the large scale covergence of the warm and moist air in a surface layer, while the vertical transfer of heat and mass due to the clouds are determined from a hypothetical model of cloud convection itself. This hypothesis has been incorporated into a fluid system which consists of two layers of incompressible, homogeneous fluid of different densities and a boundary layer to represent the effect of surface friction separately. The large scale circulation of a tropical cyclone is represented by a quasi - balanced and circularly symmetric vortex and the transverse circulation in the vortex is induced by internal and surface friction by the effect of hypothesized cloud convection. The results of linearized perturbation analysis of Ooyama's model are quite reasonable. However the results of numerical integration indicated need of improvement in some part of the model for representation of fully developed tropical cyclone.

Another model with elaborate hypothesis on the implicit cumulus convection was proposed by H.L. Kuo (1965). The numerical integration of this model appeared to produce a steady state but the computed cyclones were quite unlike those observed in nature.

Rosenthel and Kauss in 1968, proposed a linear model with multilevel vertical resolution in which the thermodynamic equation is applied at two levels. The Eigen value solutions for conditional and unconditional heating were obtained. For unconditional heating, the ratio of upper to lower troposphere heating (γ), played a important role in determining (i) whether or hot disturbances can grow (ii) The shape of the growth rate spectrum and the structure of growing distrubances. But the resulting disturbances from this model show thermal structures which are dissimilar to those of observed tropical cyclones.

Ooyama in 1969 improved his earlier model by incorporating many improvements e.g. parameterization of implicit convective clouds. The parameter η representing a measure of instability of deep convection and previously considered to be constant, is now variable and takes into account the stabilizing effect of the warm core as well as the variation of equivalent potential temperature in the boundary loyer. The general behaviour of the simulated cyclone by this model is in good agreement with the typical behaviour of real tropical cyclones in the formative deepening (immature) and mature stages. There is considerable variation of cyclone intensity in response to small changes in the sea surface temperature, as well as the rapid decay of the cyclone at simulated rainfall. This also compares well with available estimates for observed tropical cyclones.

The Ooyama's 1969 model is based on the assumption of axisymmetry, hence it is not possible to consider the movement of the cyclone centre or to investigate the interaction of the cyclone with the synoptic environment. Further more, the limited accuracy of various approximations adds to uncertainty in the quantitative verification of the model performance.

Rosenthel in 1970, constructed another "7 level primitive equation model" which included an explicit water-vapour cycle. This model was able to simulate convection originating at higher levels and non-convective precepitation, but lacked in prediction of vertical profile of radial motion in the upper tropospheric outflow layer.

More recently Chan and Williams (1987) have examined in more detail the underlying barotropic dynamics of tropical cyclone motion. The dynamical basis of their model is the conservaion of absolute vorticity, which is mathematically expressed by the non-divergent barotropic vorticity equation. This model shows that the non linear advection terms in the barotropic vorticity equation are required for significant displacement of the vortex centre and also demonstrates that larger storms have greater motion or 'beta drift' in no uniform flow experiment.

Among the latest works, most important is due to Kerry Emanuel of MIT. Prof. Emanuel (1989) constructed a simple, balanced, axisymmetric model, which is similar to Ooyama's (1969), but is phrased in Schubert and Hack's potential radius coordinates. In this model cumulus updraft mass flux depends simply and directly on the buoyancy (on angular momentum surfaces) of lifted sub cloud layer air and is not explicitly constrained by moisture convergence. The downdraft mass flux is equal to the updraft flux multiplied by $(1-\epsilon)$ where ϵ is the precipitation efficiency. The complete spectrum of convective clouds in nature has been represented by two extremes: deep clouds with a precipitation efficiency of one and shallow non-precipitating clouds. In the crude vertical structure of the model, shallow clouds have the same thermodynamic effect as precipitation induced down drafts. This model challanges the idea that tropical cyclone develops from a linear instability together with the absence of a significant reservoir of convective energy in the tropics and this led Emanuel to propose that tropical cyclones result from a "finite amplitude instability" involving the feed back between the cyclone and the wind-induced evaporation. This model proves the "finite-amplitude instability" mechanism for tropical cyclones.

Jong-Jin Baik, Mark Demaria and Sethuraman (1991) introduced a new convective parameterization scheme proposed by Betts in a tropical cyclone model, written with axisymmetric polar coordinates in the horizontal and σ coordinate in the vertical on a "f" plane. This model is called Hack and schubert Model. The new convective scheme of Betts, relaxes the vertical temperature and moisture field towards observed quasi-equialibrium thermodynamic structures. Two separate schemes have been used for deep and shallow convection. Although this scheme does not explain the detailed physical interaction between the cloud and its environment, it gives more realistic convective heating and moistening in the vertical because of the use of reference profiles based on observations. Incorporation of this scheme in ECMWF global forcast model / Hack-Schubert Model shows a significant improvement in the tropical mean flow compared with the operational Kuo-Scheme.

REFERENCES

1. K. Ooyama, J. Atmos. Sci 26, 3-39, 1969.

2. - J.J. Baik, M. DeMaria and S. Raman, Mon. Wea. Rev. 118, 513-528, (1990)

- A.K. Betts, J. Atmos. Sci, 39, 1484-1505, 1982.

  - J.Roy. Meteor. Soc. 112, 677-691, 1986.

- G.F. Carrier, A.L. Hammond and O.D. George, J. Flu. Mech., 47, 145-170, 1971.

- K.A. Emanuel, J. Atmos. Sci. 46, 3431-3456, 1989.

  -J. Atmos. Sci, 45, 1143-1155, 1988.

- M. Fiorino and R.L. Elseberry, J. Atmos. Sci, 46, 975-990, 1989

- H.L.Kuo, J. Atmos. Sci, 22, 40-63, 1965.

- J.S. Malkus and H. Riehl, Tellus, 12, 1-20, 1960.

- E Palmen and H. Riehl, J. of Met., 14, 150-159, 1957.

- S.L. Rosenthel and Walter J. Kauss, Mon.Wea. Rev. , 96, 858-866, 1968.

- S.L. Rosenthel, Mon. Wea. Rev, 98, 643-663, 1970.

- J.J. Hack and W.H. Schubert, J. Atmos. Sci. , 43, 1559-1573, 1986.

APPENDIX - A

We know that equivalent potential temperature $\overline{\theta}_E$ is given by

$$\overline{\theta}_E = \overline{\theta}\ e^{\alpha \overline{q}_s}$$

Here $\alpha = \dfrac{L}{C_p\ T}$ and all terms have got their usual meaning.

$$\ell n\overline{\theta}_E = \ell n\overline{\theta} + \alpha\overline{q}_s$$

or $$d\ \ell n\overline{\theta}_e = d\ \ell n\overline{\theta} + d(\alpha q_s)$$

or $$\frac{d\ \ell n\overline{\theta}_E}{d\ \ell n\overline{\theta}} = 1 + \frac{d\ (\alpha q_s)}{d\ (\ell n\theta)}$$

or $$1 - \frac{d\ \ell n\overline{\theta}_E}{d\ \ell n\overline{\theta}} = - \frac{d\ (\alpha q_s)}{d\ (\ell n\theta)}$$

or $$\left[1 - \frac{d\ \ell n\overline{\theta}_E}{d\ \ell n\theta}\right]_2 = -\left[\frac{d\ (\alpha q_s)}{d(\ell n\theta)}\right]_2$$

$$= \left[-\frac{d(\alpha\overline{q}_s)}{d\ \theta}\quad\frac{d\ \overline{\theta}}{d\ \ell n\theta}\right]_2$$

$$= -\left[\overline{\theta}\frac{d\ (\alpha\overline{q}_s)}{d\ \theta_2}\right]_2$$

$$= \left[-\ \alpha\overline{\theta}\quad\frac{d\ q_s}{d\theta}\right]_2$$

$$= -\frac{L}{2}\left[\frac{\overline{\theta}_3}{C_p\ \overline{T}_3} + \frac{\overline{\theta}_1}{C_p\ \overline{T}_1}\right]\frac{(q_{s3} - q_{s1})}{\theta_3 - \theta_1}$$

$$= \frac{L}{2}\left[\frac{\overline{\theta}_3}{C_p\ T_3} + \frac{\overline{\theta}1}{C_p T_1}\right]\frac{(q_{s3} - q_{s1})}{\theta_1 - \theta_3}$$

APPENDIX - B

SYMBOLS USED IN PROGRAM

MU = μ = Fraction of mean saturated humidity

N = The number of terms used in infinite continued fraction

K = κ = 1.1

Z = λ a

K_0 = Modified Bessel function of order 0

K_1 = Modified Bessel function of order 1

LA = λ_a

U = σ/K

Up = $\dfrac{\sigma/K + 1.5}{\sigma/K + 1}$

S = $[-1 + \kappa\ \mu .\ Up]^{0.5}$

SZ = S . Z

Jj = FNJ (Z, SZ) = J_1/J_0

Fj = S . Jj - K_1/K_0

```
10  ! CYC3; PROG FOR EIGENVALUES IN THE CHARNEY-
20  !       ELIASSEN MODEL
30    INPUT "Mu",Mu
40    INPUT "N",N
50    INPUT " Z,K0,K1",Z,K0,K1
60    INPUT "U0, U1",U0,U1
70    FOR U=U0 TO U1 STEP (U1-U0)/10
80      K=1.1
90      Kmu=K*Mu
100     La=Z*((U+1)/(U+.5))^.5
110     Up=(U+1.5)/(U+1)
120     S=(-1+Kmu*Up)^(-.5)
130     Sz=S*Z
140     Jj=FNJ(Z,Sz)
150     Fj=S*Jj-K1/K0
160     PRINT "FJ,LA,U,MU,Z,",Fj,La,U,Mu,Z
170   NEXT U
180   GOTO 60
190   END
200   ! XXXXXXXXXXXXXXXXXXXX
210   DEF FNJ(Z,Sz)
220     IF Sz<=1 THEN
230       Sz2=Sz*Sz/4
240       T0=Sz2/((N+1)*(N+2))
250       FOR I=N TO 1 STEP -1
260         T1=(Sz2/(I*(I+1)))/(1-T0)
270         T0=T1
280       NEXT I
290       J=(Sz/2)/(1-T0)
300       RETURN J
310     ELSE
320       T0=2*(N+1)/Sz
330       FOR I=N TO 1 STEP -1
340         T1=(2*I)/Sz-1/T0
350         T0=T1
360       NEXT I
370     END IF
380     J=1/T0
390     RETURN J
400   FNEND
```

REFERENCES

1. J.G. Charney and A. Eliassen, J. Atmos. Sci. 21, 68-75 1964

2. See for example, J.R. Holton, An Introduction to Dynamic Meteorology (Academic, N.Y.) 1972.

   G.J. Haltiner and F.L. Martinet, Dynamic and Physical Meteorology (McGraw Hill, N.Y.) 1957.

   S.L. Hess, Introduction to Theoretical Meteorology (Holt, RineHart and Winston, N.Y.) 1959.

3. C. Kittel and W.D. Knight, Mechanics (Tata McGraw Hill, New Delhi) 1965.

4. K. Ooyama, J. Atmos. Sci 26, 3-39, 1969.

5. - J.J. Baik, M. DeMaria and S. Raman, Mon. Wea. Rev. 118, 513-528, (1990)

   - A.K. Betts, J. Atmos. Sci, 39, 1484-1505, 1982.

     - J.Roy. Meteor. Soc. 112, 677-691, 1986.

   - G.F. Carrier, A.L. Hammond and O.D. George, J. Flu. Mech., 47, 145-170, 1971.

   - K.A. Emanuel, J. Atmos. Sci. 46, 3431-3456, 1989.

     -J. Atmos. Sci, 45, 1143-1155, 1988.

   - H.L.Kuo, J. Atmos. Sci, 22, 40-63, 1965.

   - J.S. Malkus and H. Riehl, Tellus, 12, 1-20, 1960.

   - E Palmen and H. Riehl, J. of Met., 14, 150-159, 1957.

   - S.L. Rosenthel and Walter J. Kauss, Mon.Wea. Rev. , 96, 858-866, 1968.

   - S.L. Rosenthel, Mon. Wea. Rev, 98, 643-663, 1970.

   - J.J. Hack and W.H. Schubert, J. Atmos. Sci. , 43, 1559-1573, 1986.