CATALAN NUMBERS AND RELATED RESULTS

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Introduction

The techniques of counting in solving combinatorial problems involving determination of the number of trees or of the sets of a given type have been in use for a long time. The first ever treatise on combinatorics - 'Disertatio de Arte Combinatoria' is due to Leibnitz and dates back to 1666. It deals essentially with the configuration which arises everytime when some objects are distributed according to some predetermined constraints.

As increasingly complex configurations came up for considerations, researchers became more and more interested in the actual counting process. The use of generating functions in this connection came to be recognised as highly useful. The idea dates back to 1812 and is due to Laplace [13]. In the present work this also has played an important role in our studies on generalized catalan numbers and results related to them. Catalan proposed these numbers first in [5] and many others have worked on them since then. Knuth for example has shown that the Catalan number C_n is exactly equal to the number of binary trees with n vertices.

Chapter I includes basic definitions and well known results which are used and elaborated upon in later chapters. In Chapter II the extended notions of Catalan numbers including the one suggested by Shapiro [27] are discussed. Some of the results related to this are extended next. And, finally, using a still more generalized form of Catalan number the results are extended further in Chapter III.

Chapter I

Preliminaries and Background Material

1.1 Generating Function and Combinatorial Identity

Definition. A formal power series,

$$A(t) = a_o + a_1 t + a_2 t^2 + \dots + a_n t^n \dots \qquad (1.1.1)$$

representing a sequence $\{a_i\}_{i \geq o}$ where a_i is a real number for each i, is called the generating function of the sequence.

The exponential generating function of the same sequence $\{a_i\}_{i \geq o}$ is defined to be

$$E(t) = a_o + a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} \dots + a_n \frac{t^n}{n!} \dots \qquad (1.1.2)$$

Generating functions have been useful in unifying the discussions on polynomials. This fact is evident from the works of Sheffer [28], Brenke [3], Rainville [22], Huff [9], Truesdell [30], Palas [20], Boas and Buck [2], Zeitlin [31] and Mittal [17, 18] and others.

It is relevant to mention about the combinatorial identities at this juncture. These concern the enumeration of ways in which a given number of objects can be arranged according to specified rules. They arise naturally in the study of generating functions and recurrence relations. In the present work they have been used extensively.

1.2 Some Operational Formulae.

Mittal [15,16,17] defined the operator T_k, where $T_k = x\,(k + xD)$, k is a constant and D is the differential operator. It is easily seen that

$$T_k^n\{x^r\} = (r+k)_n \; x^{r+n} \tag{1.2.1}$$

where

$$(r+k)_n \quad = \quad (r+k)(r+k+1) \; \ldots \; (r+k+n-1) \tag{1.2.2}$$

and n is a positive integer.

The following lemmas are due to Mittal. They have been used in this thesis in obtaining proofs of Touchard's result [29] and that of Gould [8] in the next section. They have also been used for deriving generating functions and recurrence relations for Generalized Catalan Number in chapter two and chapter three.

1. Lemma

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \; T_{a+1}^n\{x^b \; f(x)\} = x^b(1-xt)^{-a-b-1} \; f\left[\frac{x}{1-xt}\right] . \tag{1.2.3}$$

2. Lemma

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \; T_{a+n}^n\{x^b \; f(x)\} = x^b(1-xt)^{-1/2} \left[\frac{2}{1+\sqrt{(1-4xt)}}\right]^{a+b-1} f\left[\frac{2x}{1+\sqrt{(1-4xt)}}\right] . \tag{1.2.4}$$

3. Lemma

$$x \sum_{n=o}^{\infty} \frac{t^n}{n!} T^{n-1}_{a+mn+1}\{a\ f(x) + xf'(x)\} = (1+v)^a f(x(1+v))$$
(1.2.5)

where $v = xt(1+v)^{m+1}$, a and m are constants and prime denotes differentiation with respect to x.

4. Lemma

$$\sum_{n=o}^{\infty} \frac{t^n}{n!} T^n_{a+(m-1)n+1}\{f(x)\} = \frac{(1+v)^{a+1}}{(1-(m-1)v} f[x(1+v)]$$
(1.2.6)

where $v = xt(1+v)^m$ and m is a constant.

1.3 Catalan Numbers and Related Identities

While finding the number of ways of evaluating the product of n factors (in fixed order) by successive multiplications operating always on two adjacent factors, Catalan [5] used the Catalan Number.

Definition. A catalan number C_n is defined for non negative integral values of n by the rule

$$C_n = (n+1)^{-1} \binom{2n}{n} . \qquad (1.3.1)$$

A large body of research material is now available on the subject - some 450 papers have already been published. The bibliographies of Alter [1], Brown [4] and Gould [8] give an excellent account of the literature. Lois Comtet [14] and E. Netto [19] have also solved the problem of Catalan.

Touchard [29] in 1924 proved an interesting identity involving the Catalan Numbers. We give here a different proof.

Proposition.

$$\sum_{k=o}^{n} \binom{n}{2k} 2^{n-2k} C_k = C_{n+1} \tag{1.3.2}$$

where $C_k = (k+1)^{-1} \binom{2k}{k}$.

Proof. Consider

$$\sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^k \{1\},$$

in view of $T_k^n \{x^a\} = (a+k)_n x^{a+n}$, we have

$$\sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^k \{1\} = \sum_{k=o}^{\infty} \frac{x^{2k}}{(k+1)!\ k!} (k+1)(k+2)\ldots(k+1+k-1)$$

$$= \sum_{k=o}^{\infty} \frac{x^{2k}}{k!} (k+1)_k - \sum_{k=o}^{\infty} x^{2k+2} (k+3)_k$$

and we have the result

$$\sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^k \{1\} = \sum_{k=o}^{\infty} \frac{x^k}{k!} T_{k+1}^k \{1\} - x^2 \sum_{k=o}^{\infty} \frac{x^k}{k!} T_{k+3}^k \{1\}. \tag{1.3.3}$$

Now making use of (1.2.4) in (1.3.3), we get

$$\sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^{k} \{1\} = (1-4x^2)^{-1/2} \quad [1 - (\frac{2x}{1+ \sqrt{(1-4x^2)}})^2]$$

and hence

$$\sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^{k} \{1\} = \frac{2}{1 + \sqrt{(1-4x^2)}} \tag{1.3.4}$$

Now operating by $\sum_{n=o}^{\infty} \frac{2^n}{n!} T_1^n$ on both sides of (1.3.4), and using (1.2.4), we get

$$\sum_{n=o}^{\infty} \frac{2^n}{n!} T_1^n \{ \sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^{k} \{1\} \}$$

$$= \sum_{n=o}^{\infty} \frac{2^n}{n!} T_1^n \{ \frac{2}{1+ \sqrt{(1-4x^2)}} \}$$

$$= (1-2x)^{-1} [\frac{2}{1+ \sqrt{(1- \frac{4x^2}{(1-2x)^2})}}] \tag{1.3.5}$$

and hence, we get

$$\sum_{n=o}^{\infty} \frac{2^n}{n!} T_1^n \{ \sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^{k} \{1\} \}$$

$$= (\frac{2}{1+ \sqrt{(1-4x)}})^2 \,. \tag{1.3.6}$$

Again, considering the left hand side of (1.3.6) we see that

$$\sum_{n=o}^{\infty} \frac{2^n}{n!} T_1^n \left[\sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^k \{1\} \right]$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{2^n (n+2k)!}{n!\ k!\ k!\ (k+1)} x^{n+2k}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{[n/2]} 2^{n-2k} \frac{n!}{(n-2k)!\ k!\ k!\ (k+1)} x^n$$

and hence we obtain

$$\sum_{n=o}^{\infty} \frac{2^n}{n!} T_1^n \left[\sum_{k=o}^{\infty} \frac{x^k}{(k+1)!} T_{k+1}^k \{1\} \right]$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{[n/2]} \binom{n}{2k} 2^{n-2k} C_k x^n \qquad (1.3.7)$$

where $C_k = (k+1)^{-1} \binom{2k}{k}$.

Again, since

$$\sum_{n=o}^{\infty} C_{n+1} x^n = \sum_{n=o}^{\infty} \frac{1}{n+2} \binom{2n+2}{n+1} x^n$$

$$= \sum_{n=o}^{\infty} \frac{x^n}{n!} (n+2)_n - \sum_{n=o}^{\infty} \frac{x^{n+1}}{n!} (n+4)_n$$

$$= \sum_{n=o}^{\infty} \frac{1}{n!} T_{n+2}^n \{1\} - x \sum_{n=o}^{\infty} \frac{1}{n!} T_{n+4}^n \{1\}$$

$$= (1-4x)^{-\frac{1}{2}} \left[\frac{2}{1+\sqrt{(1-4x)}}\right] - x(1-4x)^{-\frac{1}{2}} \left[\frac{2}{1+\sqrt{(1-4x)}}\right]^3$$

we get

$$\sum_{n=0}^{\infty} C_{n+1}\, x^n = \left(\frac{2}{1+\sqrt{(1-4x)}}\right)^2 . \tag{1.3.8}$$

It may be noted that we have made use of Lemma (1.2.4) in the derivation of (1.3.8).

Now, from (1.3.6), (1.3.7) and (1.3.8) we conclude that

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \binom{n}{2k} 2^{n-2k} C_k\, x^n = \sum_{n=0}^{\infty} C_{n+1}\, x^n$$

and hence comparing the coefficients of x^n, we finally get

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} 2^{n-2k} C_k = C_{n+1} . \tag{1.3.9}$$

Other proofs are due to Riordon [23], Izbecki [10], Shapiro [26], Donaghey [6] and Touchard [29]. While proving Touchard's identity, Riordon [23] posed the problem : 'what is the number of Catalan paranthesis of n factors with k nests?' and solved the problem by obtaining a generating function for C_n. The number of Catalan paranthesis turned out to be

$$C_{nk} = \sum_{k=1}^{n} \binom{n-1}{2k-2} 2^{n-2k} C_{k-1} . \tag{1.3.10}$$

In the year 1976, Gould [8] established the following general identity

$$\sum_{k=o}^{[n/2]} \binom{n}{2k} 2^{n-2k} A(k) = A(n) \qquad (1.3.11)$$

where $\quad A(k) = \binom{2k}{k}$.

Subsequently he proved [8] a more general result which contained (1.3.9) and (1.3.11) as special cases. Here we give a different proof of the result.

Proposition.

$$\sum_{k=o}^{[n/2]} \binom{n}{2k}\binom{2k}{k} 2^{n-2k} R(k) = \binom{2n+2r}{n} \qquad (1.3.12)$$

where

$$R(k) = 1 \text{ if } r = 0$$

$$= \frac{(n+1)(n+2) \ldots (n+r)}{(k+1)(k+2) \ldots (k+r)} \text{ if } r \geq 1 .$$

Proof : Consider

$$\sum_{n=o}^{\infty} \sum_{k=o}^{[n/2]} \binom{n}{2k}\binom{2k}{k} R(k)\, 2^{n-2k} x^n$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{[n/2]} \binom{n}{2k}\binom{2k}{k} \frac{k!\,(n+r)!}{n!\,(k+r)!} 2^{n-2k} x^n$$

where r is a constant, we have

$$\sum_{n=o}^{\infty} \sum_{k=o}^{[n/2]} \binom{n}{2k}\binom{2k}{k} \frac{k!\,(n+r)!}{n!\,(k+r)!}\, 2^{n-2k}\, x^n$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{(n + 2k + r)!}{n!\; k!\; (k + r)!}\, 2^n\, x^{n+2k}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{1}{n!\; k!}\, (k+r+1)_{n+k}\, 2^n\, x^{n+2k}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{1}{n!\; k!}\, (k+r+1)_k\, (2k+r+1)_n\, 2^n\, x^{n+2k}$$

$$= \sum_{n=o}^{\infty} \frac{2^n}{n!}\, T^n_{r+1}\Big[\sum_{k=o}^{\infty} \frac{x^k}{k!}\, T^k_{k+r+1}\{1\}\Big] \tag{1.3.13}$$

$$= \sum_{n=o}^{\infty} \frac{2^n}{n!}\, T^n_{r+1}\Big\{(1-4x^2)^{-\frac{1}{2}} \Big[\frac{2}{1+\sqrt{(1-4x^2)}}\Big]^r\Big\}$$

$$= (1-4x)^{-\frac{1}{2}} \Big[\frac{2}{1+\sqrt{(1-4x)}}\Big]^{2r} \tag{1.3.14}$$

where r is a constant. Again it may be noted that we have used (1.2.4) and (1.2.3) in deriving (1.3.13) and (1.3.14).

Using again (1.2.4), we get from (1.3.14) the result

$$\sum_{n=o}^{\infty} \sum_{k=o}^{[n/2]} \binom{n}{2k}\binom{2k}{k} R(k)\, 2^{n-2k}\, x^n$$

$$= \sum_{n=o}^{\infty} \frac{1}{n!}\, T^n_{2r+1+n}\{1\}$$

$$= \sum_{n=o}^{\infty} \binom{2n+2r}{n}\, x^n .$$

Comparing coefficients of x^n on both sides, we get

$$\sum_{k=o}^{[n/2]} \binom{n}{2k}\binom{2k}{k} R(k)\, 2^{n-2k}$$

$$= \binom{2n+2r}{n} \tag{1.3.15}$$

where r is a constant.

It is easy to see that for r = 0 the identity in the proposition above reduces to (1.3.11) and for r = 1, reduces to (1.3.9).

Catalan sequence has been obtained while solving many other problems of combinatorics. Polya [21] in 1954 posed and solved the problem of finding the number D_n of different ways of disecting a convex polygon of n sides by n-3 diagonals into n-2 triangles. He used the recurrence relation

$$C_n = C_1\, C_{n-1} + C_2\, C_{n-2} \dots C_{n-1}\, C_1 . \tag{1.3.16}$$

Lafer and Long [12] showed that

$$D_n = \frac{1}{n-1}\binom{2n-4}{n-2} \tag{1.3.17}$$

and then D_n is same as C_{n-2}. Lafer and Long [12] gave both inductive and deductive proofs of Polya's problem. They showed that

$$
\begin{aligned}
D_3 &= 1 \\
D_4 &= 1 + 1 \\
D_5 &= 1 + 2 + 2 \\
D_6 &= 1 + 3 + 5 + 5 \\
D_7 &= 1 + 4 + 9 + 14 + 14 \\
D_8 &= 1 + 5 + 14 + 28 + 42 + 42 \\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{aligned}
\tag{1.3.18}
$$

It may be observed that the first two diagonals are Catalan sequences and are identical.

A somewhat similar array was obtained by Finucan [7]. He defined

$$
\begin{aligned}
{}^{n}F_{h} &= \binom{n+h-1}{h} - \binom{n+h-1}{h-1} \\
&= \frac{(n+1)(n+2)\ \ldots\ (n+h-1)(n+h)}{h!}
\end{aligned}
\tag{1.3.19}
$$

and obtained the following array. It is similar to that of Lafer and Long's triangular array.

(1.3.20)

n \ h	0	1	2	3	4	5	6	7
1	1	0						
2	1	1	0					
3	1	2	2	0				
4	1	3	5	5	0			
5	1	4	9	14	14	0		
6	1	5	14	28	42	42	0	
7	1	6	20	48	90	132	132	0
...	...	...	...	...	...	...	...	...

The first two diagonals in (1.3.20) are Catalan sequences again and the fourth is attributed to Cayley. No other diagonal is listed with any combinatorial meaning, though the sequences (1,3,9,...) and (1,5,20,...) occur in some problems on Laplace transform.

In conclusion we may mention that D. Knuth ([11],p.388) proved that the number of ordered trees with n vertices is C_{n-1}. He also showed that C_n is the number of binary trees with n vertices. We give an outline of the proof of this below.

<u>Theorem.</u> The total number of binary trees with n vertices equals C_n, the n^{th} Catalan number.

<u>Proof.</u> Let b_n be the number of different binary trees with n nodes. From the definition of binary trees, it is apparent that $b_o = 1$ and for $n > 0$, the number of possibilities is the number of ways to put binary trees with k nodes to the left of the root and another with n-1-k nodes to the right. So

$$b_n = b_o\, b_{n-1} + b_1\, b_{n-2} \cdots b_{n-1}\, b_o \quad n \geq 1 \tag{1.3.21}$$

From this it is clear that the generating function

$$B(z) = b_o + b_1 z + b_2 z^2 + \cdots \tag{1.3.22}$$

satisfies the equation

$$z\, B(z)^2 = B(z) - 1 \tag{1.3.23}$$

Solving the quadratic equation and using the fact that $B(0) = 1$, he obtained

$$\begin{aligned} B(z) &= \frac{1}{2z}\left(1 - \sqrt{1 - 4z}\right) \\ &= \frac{1}{2z}\left(1 - \sum_{n \geq 0} \binom{1/2}{n} (-4z)^n\right) \\ &= \sum_{n \geq 0} \binom{1/2}{n+1} (-1)^n \, 2^{2n+1} \, z^n . \end{aligned}$$

We now compare coefficients of z^n in (1.3.21) and obtain

$$\begin{aligned} b_n &= \binom{1/2}{n+1} (-1)^n \, 2^{2n+1} \\ &= \frac{1}{n+1} \binom{2n}{n} . \end{aligned} \qquad (1.3.24)$$

Chapter - II

Extended Catalan Numbers

The notion of catalan numbers which was introduced by Catalan in [5] was broadened later by Shapiro [27]. In a series of papers [25,26,27] he has considered their properties in detail. The motivation for this consideration was graph theoretic.

Definition.

A finite sequence of pairs $v_k = (a_k, b_k)$, $a_n \geq 0$, $b_n \geq 0$ is called a path if the following hold

a) $v_o = (0,0)$

b) If $v_k = (a_k, b_k)$ then $v_{k+1} = (1+a_k,\ b_k)$ or $v_{k+1} = (a_k,\ 1+b_k)$

A path $(v_o, v_1, \ldots, v_n)$ is said to be of length n and the distance between $\{v_i\}_{i=o}^{n} = \{(a_i, b_i)\}_{i=o}^{n}$ and $\{w_i\}_{i=o}^{n} = \{(x_i, y_i)\}_{i=o}^{n}$ is $|a_n - x_n|$. Two paths are said to intersect if $v_i = w_i$ for some $o < i \leq n$.

Remark. It may be noted that $a_n + b_n = n = x_n + y_n$. Hence $|a_n - x_n| = |b_n - y_n|$ and consequently the distance between the paths $\{v_i\}_{i=o}^{n}$ and $\{w_i\}_{i=o}^{n}$ could be defined as $|b_n - y_n|$.

One may observe that a pair of paths of length n at distance k can be extended to four pairs of paths of length n+1: one pair at distance k+1, two pairs at distance k, and one pair at distance (k-1). If B_{nk} is the number of pairs of non intersecting paths of length n and distance k then it is not difficult to derive the recurrence relation from the above observation. Then

$$B_{nk} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1} \quad \text{(Shapiro [27])}.$$

and $B_{no} = o = B_{n,n+m}$, $m \geq 1$ are boundary conditions. Shapiro [27] found

$$B_{nk} = \frac{k}{n} \binom{2n}{n-k} \tag{2.1.1}$$

For $k = 1$, $B_{n1} = C_n = \frac{1}{n} \binom{2n}{n-1} = \frac{1}{n} \binom{2n}{n+1}$

where n and k are positive integers in above formula (2.1.1). Tabulation of B_{nk} yields the following triangular array which Shapiro [27] named as catalan triangle.

n \ k	1	2	3	4	5	6
1	1					
2	2	1				
3	5	4	1			
4	14	14	6	1		
5	42	48	27	8	1	
6	132	165	110	44	10	1
...	...	...	...	...	...	...

(2.1.2)

In what follows we introduce the notion of extended Catalan number and extend the results of Shapiro [27].

<u>Definition</u>. The extended catalan number $B_{nk}^{(a)}$ is defined by the rule

$$B_{nk}^{(a)} = \frac{k+a}{n+a}\binom{2n+2a}{n-k} \tag{2.1.3}$$

$$= \frac{(k+a)}{n+a}\,\frac{(2n+2a)(2n+2a-1)\ \ldots\ (n+2a+k+1)}{(n-k)!} \tag{2.1.4}$$

where $n-k \geq o$, $n > o$ and a is a real number and $a \neq -n$

For $a = -n$, we define

$$B_{nk}^{(-n)} = \frac{2(k+a)(2n+2a-1)\ \ldots\ (n+2a+k+1)}{(n-k)!}$$

$$= \lim_{a \to -n} B_{nk}^{(a)} .$$

<u>Remarks</u>.

1. We may use the above definition to write

$B_{oo}^{(a)} = \frac{a}{a} = 1.$ and we may define $B_{oo}^{(a)} = \lim_{a \to o} B_{oo}^{(a)} = 1.$

2. For $a = o$, $B_{nk}^{(o)}$ coincides with the notion of Catalan Number introduced by Shapiro [27].

3. For $a = .5$ and $a = 1$, $B_{nk}^{(a)}$ coincides with the notion of Ballot numbers $g_{(2n,2m)}$ and $g_{(2n+1,\ 2m+1)}$ determined by Knuth [11] respectively where

$$g_{(2n,2m)} = \frac{2m+1}{2n+1}\binom{2n+1}{n-m}$$

$$g_{(2n+1,2m+1)} = \frac{2m+2}{2n+2}\binom{2n+2}{n-m}, \quad \text{n and m being positive integers.}$$

We consider next arithmatic properties of the extended Catalan Numbers and the associated results are shown to be direct extension of the results of Shapiro [27].

We shall evaluate $\sum_{k=1}^{n} B_{nk}^{(a)}$.

<u>Proposition.</u> $$\sum_{k=1}^{n} B_{nk}^{(a)} = \binom{2n+2a-1}{n-1} \qquad (2.1.5)$$

<u>Proof</u>. Consider the power series

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} B_{nk}^{(a)} x^{n}$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} B_{n+k-1,k}^{(a)} x^{n+k-1}$$

$$= \sum_{n=o}^{\infty}\sum_{k=1}^{\infty} B_{n+k,k}^{(a)} x^{n+k}$$

$$= \sum_{n=o}^{\infty}\sum_{k=1}^{\infty} \frac{k+a}{n+k+a}\binom{2n+2k+2a}{n} x^{n+k}$$

$$= \sum_{n=o}^{\infty}\sum_{k=1}^{\infty} 2(k+a)\frac{1}{n!}\frac{\Gamma(2n+2k+2a)}{\Gamma(n+2k+2a+1)} x^{n+k}$$

$$= \sum_{n=o}^{\infty}\sum_{k=1}^{\infty} \frac{x^{k+1}}{n!} T_{n+2k+2a+1}^{n-1}\{2(n+a)\}$$

$$= \sum_{k=1}^{\infty} x^{k}(1+v)^{2k+2a} \quad \text{where } v = x(1+v)^{2} \qquad (2.1.6)$$

$$= (1+v)^{2a} \sum_{k=1}^{\infty} x^k(1+v)^{2k}$$

$$= (1+v)^{2a} \sum_{k=1}^{\infty} v^k$$

$$= \frac{x\ (1+v)^{2a+2}}{(1-v)}$$

$$= x \sum_{n=o}^{\infty} \frac{1}{n!} T^n_{n+2a+2} \{1\} \qquad (2.1.7)$$

$$= x \sum_{n=o}^{\infty} \frac{1}{n!} (n+2a+2)_n\ x^n$$

$$= x \sum_{n=o}^{\infty} \binom{2n+2a+1}{n} x^n$$

$$= \sum_{n=o}^{\infty} \binom{2n+2a+1}{n} x^{n+1}$$

$$= \sum_{n=1}^{\infty} \binom{2n+2a-1}{n-1} x^n . \qquad (2.1.8)$$

Lemma (1.2.5) and Lemma (1.2.6) has been used above in deriving (2.1.6) and (2.1.7) respectively.

Comparing the coefficients of x^n in (2.1.8) we conclude

$$\sum_{k=1}^{n} B^{(a)}_{nk} = \binom{2n+2a-1}{n-1} . \qquad (2.1.9)$$

Remark. Putting a = o in (2.1.9), we get

$$\sum_{k=1}^{n} B_{nk}^{(o)} = \binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n} \qquad (2.1.10)$$

which is due to Shapiro [27].

Corollary. The generating function of $B_{nk}^{(a)}$ is determined from

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(a)} x^n = (1-4x)^{-\frac{1}{2}} \left(\frac{2}{1+\sqrt{(1-4x)}}\right)^{2a-1} \qquad (2.1.11)$$

Proof.

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(a)} x^n = \sum_{n=o}^{\infty} \sum_{k=o}^{n} \frac{k+a}{n+a} \binom{2n+2a}{n-k} x^n$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{x^{n+1}}{n!} T_{n+2k+2a+1}^{n-1} \left\{2k+2a\right\}$$

$$= (1+v)^{2a} \left[1-x(1+v)^2\right]^{-1}$$

$$= \frac{(1+v)^{2a}}{1-v}, \quad \left[v = x(1+v)^2\right] \qquad (2.1.12)$$

$$= (1-4x)^{-\frac{1}{2}} \left[\frac{2}{1+\sqrt{(1-4x)}}\right]^{2a-1}.$$

Proposition.

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(a)} x^{n-k} = \sum_{k=o}^{\infty} \left(\sum_{n=1}^{\infty} C_n x^{n-1}\right)^{k+a} \qquad (2.1.13)$$

where C_n is the nth Catalan Number.

Proof. Consider

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(a)} \; x^{n-k}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} 2(k+a)(n+2k+2a+1)_{n-1} \; x^{n}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{x}{n!} \; T_{n+2k+2a+1}^{n-1}\{2k+2a\}$$

$$= \sum_{k=o}^{\infty} (1+v)^{2k+2a} \qquad (2.1.14)$$

which we get using lemma (1.2.5), where $v = x(1+v)^2$, and a is a constant.

Since $v = x(1+v)^2$

$$1+v = 1+x(1+v)^2$$

and we obtain the generating function of Catalan Numbers

$$C(x) = \sum_{n=o}^{\infty} C_n \; x^n$$

$$= 1+x\, C^2(x) \qquad \text{(Riordon [23])}$$

$$= \frac{1-\sqrt{(1-4x)}}{2x} \qquad \text{(Riordon [32])}$$

$$= \frac{2}{1+\sqrt{(1-4x)}} \qquad (2.1.15)$$

we can write now

$$1+v = C(x) = \frac{2}{1+\sqrt{(1-4x)}} \qquad (2.1.16)$$

In view of (2.1.16) we see that

$$\sum_{k=o}^{\infty} (1+v)^{2k+2a} = \sum_{k=o}^{\infty} \left(\frac{2}{1+\sqrt{(1-4x)}}\right)^{2k+2a}. \tag{2.1.17}$$

Earlier we also showed (1.3.8)

$$\sum_{n=o}^{\infty} C_{n+1}\, x^{n} = \left(\frac{2}{1+\sqrt{(1-4x)}}\right)^{2}. \tag{2.1.18}$$

Making use of (2.1.17) and (2.1.18) we get from (2.1.14)

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(a)}\, x^{n-k} = \sum_{k=o}^{\infty} \left(\sum_{n=o}^{\infty} C_{n+1}\, x^{n}\right)^{k+a}$$

$$= \sum_{k=o}^{\infty} \left(\sum_{n=1}^{\infty} C_{n}\, x^{n-1}\right)^{k+a}. \tag{2.1.19}$$

Remark.

Since for $a = o$, $k = 1$ we have

$$B_{n1}^{(o)} = \frac{1}{n}\binom{2n}{n-1} = C_n$$

where C_n is nth Catalan Number, we conclude,

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(a)}\, x^{n-k} = \sum_{k=o}^{\infty} \left(\sum_{n=1}^{\infty} B_{n1}^{(o)}\, x^{n-1}\right)^{k} \tag{2.1.20}$$

which is similar to Shapiro's result (Rogers [24]).

Proposition.

$$B_{nk}^{(a)} = \sum_{j=1}^{n-k+1} C_j\, B_{n-j,k-1}^{(a)} \tag{2.1.21}$$

where j is a positive integer.

<u>Proof</u>. Consider

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} \sum_{j=1}^{n-k+1} C_j \, B^{(a)}_{n-j,k-1} \, x^{n-k}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \sum_{j=1}^{n+1} C_j \, B^{(a)}_{n+k-j,k-1} \, x^{n}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \sum_{j=o}^{n} C_{j+1} \, B^{(a)}_{n+k-j-1,k-1} \, x^{n}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \sum_{j=o}^{\infty} C_{j+1} \, B^{(a)}_{n+k-1,k-1} \, x^{n+j}$$

$$= \sum_{j=o}^{\infty} \frac{1}{j+2} \binom{2j+2}{j+1} x^j \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{k-1+a}{n+k-1+a} \binom{2n+2k-2+2a}{n} x^n$$

$$= \sum_{j=o}^{\infty} \frac{2}{j!} x^j \, (j+3)_{j-1} \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{2(k-1+a)}{n!} (n+2k+2a-1)_{n-1} \, x^n$$

$$= \sum_{j=o}^{\infty} x \, T^{j-1}_{j+3} \; 2 \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{x}{n!} \, T^{n-1}_{n+2k+2a-1} \{2(k-1+a)\}$$

$$= (1+v)^2 \sum_{k=o}^{\infty} (1+v)^{2k+2a-1} \tag{2.1.22}$$

where $v = x(1+v)^2$, a being a constant. Here Lemma (1.2.5) had been used for deriving (2.1.22) we have from (2.1.22),

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} \sum_{j=1}^{n-k+1} C_j \, B^{(a)}_{n-j,k-1} \, x^{n-k}$$

$$= \sum_{k=o}^{\infty} (1+v)^{2k+2a} . \tag{2.1.23}$$

Since we have already proved in (2.1.14)

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(a)} x^{n-k} = \sum_{k=o}^{\infty} (1+v)^{2k+2a} . \tag{2.1.24}$$

we have from (2.1.23) and (2.1.24)

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} \sum_{j=1}^{n-k+1} C_j B_{n-j,k-1}^{(a)} x^{n-k}$$
$$= \sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(a)} x^{n-k} . \tag{2.1.25}$$

Now comparing the coefficients of x^{n-k} in (2.1.25) on both sides we get

$$B_{nk}^{(a)} = \sum_{j=1}^{n-k+1} C_j B_{n-j,k-1}^{(a)} \tag{2.1.26}$$

<u>Remark</u>. The result in (2.1.26) reduces to Shapiro's result [27] for a = o. We have from (2.1.26) for a = o,

$$B_{nk}^{(o)} = \sum_{j=1}^{n-k+1} C_j B_{n-j,k-1}^{(o)} .$$

The above is equivalent to the following.

$$B_{nk} = \sum_{j=1}^{n-k+1} C_j B_{n-j,k-1} \tag{2.1.27}$$

where $B_{nk} = B_{nk}^{(o)} = \frac{k}{n} \binom{2n}{n-k}$.

The identity in (2.1.27) is due to Shapiro [27].

2.2 Recurrence Relations.

Next we obtain a recurrence relation of the extended Catalan Number. It is similar to the one determined by Shapiro [27].

Proposition. $B^{(a)}_{n+1,k+1} = B^{(a)}_{nk} + 2B^{(a)}_{n,k+1} + B^{(a)}_{n,k+2}$ (2.2.1)

where $n \geq o$.

Proof. From (2.1.6), we have

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B^{(a)}_{nk} x^n = \sum_{k=o}^{\infty} x^k (1+v)^{2k+2a} \tag{2.2.2}$$

where a is a constant and $v = x(1+v)^2$. In a similar manner we can see that

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B^{(a)}_{n+1,k+1} x^n = \sum_{k=o}^{\infty} x^k (1+v)^{2k+2a+2} \tag{2.2.3}$$

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B^{(a)}_{n,k+1} x^n = \sum_{k=o}^{\infty} x^{k+1} (1+v)^{2k+2a+2} \tag{2.2.4}$$

and

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B^{(a)}_{n,k+2} x^n = \sum_{k=o}^{\infty} x^{k+2} (1+v)^{2k+4+2a} \tag{2.2.5}$$

where $v = x(1+v)^2$ and a is a constant.

In view of (2.2.2), (2.2.4) and (2.2.5), we conclude

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} [B^{(a)}_{nk} + 2B^{(a)}_{nk+1} + B^{(a)}_{nk+2}] x^n$$

$$= (1+v)^{2a} \sum_{k=o}^{\infty} (1+v)^{2k+2} x^k \left[\frac{1}{(1+v)^2} + 2x + x^2(1+v)^2\right]$$

$$= (1+v)^{2a} \sum_{k=o}^{\infty} (1+v)^{2k+2} x^k \left[\frac{1}{1+v} + x(1+v) \right]^2$$

$$= (1+v)^{2a} \sum_{k=o}^{\infty} (1+v)^{2k+2} x^k \left[\frac{1+x(1+v)^2}{1+v} \right]^2. \qquad (2.2.6)$$

And since $v = x(1+v)^2$, we have from (2.2.6)

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} \left[B_{nk}^{(a)} + 2B_{n,k+1}^{(a)} + B_{n,k+2}^{(a)}\right] x^n$$

$$= \sum_{k=o}^{\infty} (1+v)^{2k+2a+2} x^k$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{n+1,k+1}^{(a)} x^n \qquad (2.2.7)$$

Comparing the coefficients of x^n on both the sides of (2.2.7), the identity in (2.2.1) follows.

<u>Remark</u>. The result in (2.2.1) is an extension of the result of Shapiro [27] and reduces to Shapiro's result if $a = o$ and $n \geq 1$,

$$B_{nk} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}$$

where $B_{nk} = \frac{k}{n} \binom{2n}{n-k}$.

In a similar manner we can derive the proof for following corollaries.

<u>Corollary 1</u>.

$$B_{n+1,k+1}^{(a)} = B_{n+1,k+1}^{(a-1)} + 2B_{n,k+1}^{(a)} + B_{n,k+2}^{(a)}. \qquad (2.2.8)$$

Corollary 2.

$$B^{(a)}_{n+2,k+2} = B^{(a)}_{n+1,k+1} + 2B^{(a+1)}_{n,k+1} + B^{(a)}_{n+1,k+3}\,. \qquad (2.2.9)$$

Corollary.3.

$$B^{(a)}_{n+2,k+2} = B^{(a)}_{n+1,k+1} + 2B^{(a)}_{n+1,k+2} + B^{(a+1)}_{n,k+2}\,. \qquad (2.2.10)$$

Corollary.4.

$$B^{(a)}_{n+1,k+1} = B^{(a+1)}_{n,k} \qquad (2.2.11)$$

and $$B^{(a)}_{n,k} = B^{(a-1)}_{n+1,k+1}\,.$$

2.3 Triangular Arrays

The extended notion of Catalan Numbers was used by Shapiro [27] in setting up a triangular form. For a particular value of k all the Catalan Numbers B_{nk} were arranged in one column as follows.

n \ k	1	2	3	4	5	6
1	1					
2	2	1				
3	5	4	1			
4	14	14	6	1		
5	42	48	27	8	1	
6	132	165	110	44	10	1
...	...	...	...	...	...	...

(2.3.1)

The notion of Catalan Numbers which we have used allows an extension of the above.

From $B_{nk}^{(a)} = \frac{k+a}{n+a}\binom{2n+2a}{n-k}$, one may obtain the following triangular arrays.

For a = .5, we have,

n \ k	0	1	2	3	4	5
0	1					
1	1	1				
2	2	3	1			
3	5	9	5	1		
4	14	28	20	7	1	
5	42	90	75	35	9	1
...	...	...	...	...	...	...

(2.3.2)

It is evident that the Catalan Sequence appears in the first column while rest of the columns do not appear of the Catalan Triangle developed by Shapiro [27].

For a = 1, we have

n \ k	0	1	2	3	4	5
0	1					
1	2	1				
2	5	4	1			
3	14	14	6	1		
4	42	48	27	8	1	
5	132	165	110	44	10	1
...	...	...	...	...	...	...

(2.3.3)

The array (2.3.3) is similar to Catalan Triangle developed by Shapiro [27].

Remark. However the columns in (2.3.2) and (2.3.3) appear alternatingly in Triangular tableau developed for D_n by Lafer and Long [12] in diagonal form. A similar table of numbers was obtained by Finucan [7].

We compute below the table (2.3.4) for a = - 4.

n \ k	0	1	2	3	4	5	6	7	8
0									
1									
2									
3									
4					1				
5				-1	0	1			
6			-1	-2	0	2	1		
7		-1	-4	-5	0	5	4	1	
8	-1	-6	-14	-14	0	14	14	6	1
...	...	...	...	...	...	...	...	...	...

(2.3.4)

It is evident from (2.3.4) that the columns left to 'Column of Symmetry' (k = 4) are mirror images of the columns on the right of the 'column of symmetry.' The triangular array on the right of the 'column of symmetry' is same as the Catalan Triangle developed by Shapiro [27].

For a = -5.5 we get the following triangular array.

n \ k	0	1	2	3	4	5	6	7	8	9	10	11
0												
1												
2												
3												
4												
5												
6						-1	-1					
7					-1	-1	1	1				
8				-1	-3	-2	2	3	1			
9			-1	-5	-9	-5	5	9	5	1		
10		-1	-7	-20	-28	-14	14	28	20	7	1	
11	-1	-9	-35	-75	-90	-42	42	90	75	35	9	1
...	...	...	...	...	...	...	...	...	...	...	...	...

(2.3.5)

It may be observed that the column corresponding to k = 6 and higher values constitute the Catalan triangle for a = .5. The columns corresponding to n = o,1,2,...,5 could be viewed as mirror images of the columns of the Catalan Triangle mentioned for a = .5.

Chapter III

Enlarged Class of Extended Catalan Numbers and Related Generalisations

3.1 Further Extension of the notion of Catalan Numbers.

We saw in section (2.3) that tabulation of $B_{nk}^{(a)}$ for $a = -4$ and $a = -5.5$ yielded two different tables. All of the columns in either of the tables appear however as diagonals in the results of Lafer and Long [12] and Finucan [7].

We are thus motivated to define extended Catalan Numbers in a way so that the columns in both of the tables mentioned above appear in one single table, similar to that of Lafer and Long [12] and Finucan [7].

Definition. An extended Catalan Number B_{ns} is defined by the rule

$B_{ns} = \frac{s}{2n-s} \binom{2n-s}{n}$ where n and s are (3.1.1) positive (3·1·1)
integers and $n-s \geq 0$.

Remark 1. For $s = 1$, $B_{n1} = \frac{1}{2n-1} \binom{2n-1}{n}$

$$= \frac{(2n-2)!}{n!\,(n-1)!}$$

$$= C_{n-1}$$

where C_{n-1} is the $(n-1)^{th}$ Catalan number.

2. $B_{n1} = B_{n2} = C_{n-1}, \quad n \geq 2.$

3. $B_{n+1,2} = C_n, \quad n \geq 1.$

4. one may define $B_{oo} = 0$ if necessary.

Calculation with the number B_{ns} yields the following table

n \ s	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	2	1				
4	5	5	3	1			
5	14	14	9	4	1		
6	42	42	28	14	5	1	
7	132	132	90	48	20	6	1
...	...	...	...	...	...	...	...

(3.1.2)

The above triangular array is similar to that of Lafer and Long [12] and Finucan [7]. It can also be noticed that the columns in (2.3.2) and (2.3.3) appear alternatingly in above triangular array.

Next we obtain the recurrence relation for the newly defined Catalan Number.

<u>Proposition</u>. $B_{n+1,s+1} = B_{ns} + B_{n+1,s+2}$, (3.1.3)

<u>Proof</u>. Consider

$$B_{n+1,s+1} - B_{n+1,s+2} = \frac{s+1}{2n-s+1}\binom{2n-s+1}{n+1} - \frac{s+2}{2n-s}\binom{2n-s}{n+1}$$

$$= \frac{s+1}{n+1} \frac{(2n-s)!}{n!\,(n-s)!} - \frac{s+2}{n+1} \frac{(n-s)}{2n-s} \frac{(2n-s)!}{n!}$$

$$= \frac{1}{2n-s} \binom{2n-s}{n} \left[\frac{(2n-s)(s+1)-(s+2)(n-s)!}{n+1}\right]$$

$$= \frac{1}{2n-s} \binom{2n-s}{n} \left[\frac{(n+1)s}{n+1}\right]$$

$$= B_{ns} \quad . \qquad (3.1.4)$$

We conclude from (3.1.4)

$$B_{n+1,s+1} = B_{ns} + B_{n+1,s+2} \; .$$

Corollary.

$$\sum_{s=1}^{n} B_{ns} = B_{n+1,2} = C_n \qquad (3.1.5)$$

Proof. In $B_{n+1,s+1} = B_{ns} + B_{n+1,s+2}$ we substitute $s = 1,2,\ldots n-1$ and obtain the following:

$$B_{n+1,2} = B_{n1} + B_{n+1,3} \qquad (1)$$

$$B_{n+1,3} = B_{n2} + B_{n+1,4} \qquad (2)$$

....

....

....

$$B_{n+1,n} = B_{n,n-1} + B_{n+1,n+1}. \qquad (3)$$

Adding all these n equations after cancellation we obtain

$$B_{n+1,2} = \sum_{s=1}^{n-1} B_{ns} + 1$$

$$= \sum_{s=1}^{n-1} B_{ns} + B_{nn}$$

$$= \sum_{s=1}^{n} B_{ns} \ . \qquad (3.1.16.)$$

once again

$$B_{n+1,2} = \frac{2}{2n} \binom{2n}{n+1} = C_n \ .$$

Thus we conclude

$$\sum_{s=1}^{n} B_{ns} = B_{n+1,2} = C_n.$$

<u>Proposition.</u>

The generating function for B_{ns} is given by

$$\sum_{n=1}^{\infty} \sum_{s=1}^{s=n} B_{ns}\, x^n = \frac{4x}{(1+\sqrt{(1-4x)})^2} \qquad (3.1.17)$$

<u>Proof</u>:

$$\sum_{n=1}^{\infty} \sum_{s=1}^{s=n} B_{ns}\, x^n = \sum_{n=o}^{\infty} \sum_{s=1}^{\infty} B_{n+s,s}\, x^{n+s}$$

$$= \sum_{n=o}^{\infty} \sum_{s=1}^{\infty} \frac{s}{2n+s} \binom{2n+s}{n+s} x^{n+s}$$

$$= \sum_{n=o}^{\infty} \sum_{s=1}^{\infty} \frac{s}{n!} (n+s+1)_{n-1}\, x^{n+s}$$

$$= \sum_{n=o}^{\infty} \sum_{s=1}^{\infty} \frac{x^{s+1}}{n!} T_{n+s+1}^{n-1} \{s\}$$

$$= \sum_{s=1}^{\infty} (1+v)^s x^s \quad \text{by lemma}(1.2.5)$$

(3.1.18)

and from the fact that $v = x(1+v)^2$.

$$\text{Thus, } \sum_{n=1}^{\infty} \sum_{s=1}^{s=n} B_{ns} x^n = \sum_{s=1}^{\infty} (1+v)^s x^s$$

$$= \sum_{s=1}^{\infty} \frac{(1+v)^s v^s}{(1+v)^{2s}}$$

$$= \sum_{s=1}^{\infty} \frac{v^s}{(1+v)^s} = \sum_{s=1}^{\infty} \left(\frac{v}{1+v}\right)^s$$

$$= \frac{\frac{v}{1+v}}{1 - \frac{v}{1+v}}$$

$$= v$$

$$= x(1+v)^2$$

$$= \frac{4x}{(1+\sqrt{(1-4x)}^2)} .$$

(3.1.19)

<u>Proposition</u>.

$$\sum_{n=1}^{\infty} \sum_{s=1}^{s=n} B_{ns} x^{n-s} = \sum_{s=o}^{\infty} \left(\sum_{n=1}^{\infty} C_n x^{n-1} \right)^{s/2}.$$

The proof is similar to that of (2.1.13) and is therefore omitted.

The number B_{ns} admits immediate generalization. Analogous to $B_{nk}^{(a)}$, we next define $B_{ns}^{(a)}$, an extension of the notion of Catalan Numbers which has just been introduced.

<u>Definition.</u> The extended Catalan Number $B_{ns}^{(a)}$ is defined by the rule

$$B_{ns}^{(a)} = \frac{s+a}{2n-s-a}\binom{2n-s-a}{n} \tag{3.1.10}$$

where a is a constant.

The proofs of the propositions which follow are easy and are similar to those given for analogous results established earlier. These are therefore omitted.

1. <u>Proposition.</u> $B_{n+1,s+1}^{(a)} = B_{ns}^{(a)} + B_{n+1,s+2}^{(a)}$. (3.1.11)

2. <u>Proposition.</u>

$$\sum_{n=1}^{\infty}\sum_{s=1}^{\infty} B_{ns}^{(a)} x^n = \left(\frac{2}{1+\sqrt{(1-4x)}}\right)^a \frac{4x}{(1+\sqrt{(1-4x)})^2}. \tag{3.1.12}$$

3. <u>Proposition.</u>

$$\sum_{n=1}^{\infty}\sum_{s=1}^{n} B_{ns}^{(a)} x^{n-s} = \sum_{s=o}^{\infty}\left(\sum_{n=1}^{\infty} C_{n-1}x^n\right)^{\frac{s+a}{2}}. \tag{3.1.13}$$

We now tabulate the following triangular array for a = -8.

n \ s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1							-1	0	1						
2						-1	-1	0	1	1					
3					-1	-2	-2	0	2	2	1				
4				-1	-3	-5	-5	0	5	5	3	1			
5			-1	-4	-9	-14	-14	0	14	14	9	4	1		
6		-1	-5	-14	-28	-42	-42	0	42	42	28	14	5	1	
7	-1	-6	-20	-48	-90	-132	-132	0	132	132	90	48	20	6	1
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...

(3.1.14)

The columns in left of 'column of symmetry' (for $s = 8$) appears to be mirror image of the columns on right side of it. The columns on right of the 'column of symmetry' is similar to the array defined by Lafer and Long and Finucan.

We next obtain further extension of the results already established in chapter II. It is relevant to mention that Rogers [24] developed the following notion for defining Catalan sequence.

$$C_t(n) = \frac{1}{tn+1}\binom{(t+1)n}{n} \quad n \geq 0, \quad t \geq 0. \qquad (3.1.15)$$

$C_t(n)$ clearly represents an extension of the idea of C_n. For $t = 1$, $C_1(n) = C_n$. These sequences occur in wide variety of combinatorial problems. Rogers [24] further define

$$B_t(n,m) = \sum_{r=o}^{t+1} \binom{t+1}{n} B_t\,(n-1,\ m-1+r)$$

and concluded that

$$B_{t-1}(n,m) = \frac{m+1}{n+1} \binom{t(n+1)}{n-m} \qquad (3.1.16)$$

To generalize further the result of Rogers (3.1.16), we define below the number $B_{nk}^{(p,a)}$ which subsequently generalizes the result of Shapiro (2.1.1) as well as the notion of extended Catalan number $B_{nk}^{(a)}$ which we introduced in chapter II

<u>Definition</u>. The generalized Catalan Number $B_{nk}^{(p,a)}$ is given by the rule

$$B_{nk}^{(p,a)} = \frac{k+a}{n+a} \binom{pn+pa}{n-k} \qquad (3.1.17)$$

where n and k are positive integers and p and a are constants.

<u>Remarks.</u>

1. Generalized Catalan number is an extension of the notion of $B_{nk}^{(a)}$. It reduces to $B_{nk}^{(a)}$ for $p = 2$.

$$B_{nk}^{(2,a)} = \frac{k+a}{n+a} \binom{2n+2a}{n-k}$$

$$= B_{nk}^{(a)} \quad \text{(See (2.1.3)).}$$

2. $B_{nk}^{(p,1)} = \frac{k+1}{n+1} \binom{(p+1)\,n}{n-k}$ (3.1.18)

$= B_{t-1}(n,m)$ which is due to Rogers as mentioned above.

3. For $p = 2$ and $a = 0$

$$B_{nk}^{(2,0)} = \frac{k}{n} \binom{2n}{n-k}$$

$= B_{nk}$, the extended Catalan numbers introduced by Shapiro [27].

4. For $k = 1$, $a = o$ and $p = 2$.

$$B_{n,1}^{(2,0)} = \frac{1}{n} \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$$

$= C_n$, the Catalan number as given by Catalan [5].

Notion of Generalized Catalan Number includes also that of the Ballot numbers (Knuth [11], p. 532). For $p = 2$, $a = .5$ and $p = 2$, $a = 1$ respectively t yields the Ballot numbers.

Our next aim is to establish a combinatorial identity involving the generalized Catalan Number and which is also an extension of Shapiro's result [27] we shall rely largely on the lemmas due to Mittal [15, 16, 17].

Proposition.

$$\sum_{k=o}^{n} (p-1)^k \frac{k+a}{n+a} \binom{pn+pa}{n-k} = \binom{pn+pa-1}{n} \qquad (3.1.19)$$

where p and a are constants.

Proof. Consider the following power series for obtaining the generating function of the Generalized Catalan Number $B_{nk}^{(p,a)}$ in order to prove (3.1.19).

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} (p-1)^k B_{nk}^{(p,a)} x^n$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} (p-1)^k \frac{x^{n+k}}{n!} p(k+a).((p-1)n+pk+pa+1)n-1$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} (p-1)^k \frac{x^{k+1}}{n!} T^{n-1}_{(p-1)n+pk+pa+1}\{p(k+a)\}$$

$$= \sum_{k=o}^{\infty} (p-1)^k (1+v)^{pk+pa} x^k \qquad (3.1.20)$$

where $v = x(1+v)^p$, p and a are constants.

Lemma (1.2.5) has been used in the last step.

Now in view of $v = x(1+v)^p$, we have

$$\sum_{k=o}^{\infty} (1+v)^{pk+pa} (p-1)^k x^k$$

$$= \sum_{k=o}^{\infty} \frac{(1+v)^{pk+pa}}{1-(p-1)v} (p-1)^k x^k - \sum_{k=o}^{\infty} (p-1)^{k+1} \frac{x^{k+1}(1+v)^{pk+pa+p}}{(1-(p-1)v}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} (p-1)^k \frac{x^k}{n!} T^n_{(p-1)n+pk+pa} \{1\}$$

$$- \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} (p-1)^{k+1} \frac{x^{k+1}}{n!} T^n_{(p-1)n+pk+pa+p} \{1\}$$

(3.1.21)

Now substituting (3.1.21) in (3.1.20) and using lemma (1.2.6) again, we get

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(p,a)} (p-1)^k x^n$$

$$= \frac{(1+v)^{pa}}{(1-(p-1)v)} \left[\sum_{k=o}^{\infty} (p-1)^k x^k (1+v)^{pk} \right.$$

$$\left. 1 - (p-1)x(1+v)^p \right] \qquad (3.1.22)$$

$$= \frac{(1+v)^{pa}}{(1-(p-1)v} . \qquad (3.1.22)$$

where $v = x(1+v)^p$, using lemma (1.2.6), we have from (3.1.22)

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(p,a)} x^n (p-1)^k = \sum_{n=o}^{\infty} \frac{1}{n!} T^n_{(p-1)n+pa} \{1\}$$

$$= \sum_{n=o}^{\infty} x^n \binom{pn+pa-1}{n} \qquad (3.1.23)$$

Comparing the coefficients of x^n on both the sides of (3.1.23), we get

$$\sum_{k=o}^{n} (p-1)^k B_{nk}^{(p,a)} = \binom{pn+pa-1}{n} \qquad (3.1.24)$$

<u>Remark 1.</u> The result in (3.1.24) is can be found a generalization of the result (2.1.6). For $p = 2$, $B_{nk}^{(p,a)}$ gives

$$\sum_{k=0}^{n} \frac{k+a}{n+a} \binom{2n+2a}{n-k} = \binom{2n+2a-1}{n}$$

which we have proved in chapter II.

2. The result (3.1.24) is generalization of Shapiro's result, for $a = 0$ and $p = 2$, we get Shapiro's identity. For $a = 0$, $p = 2$ we have from (3.1.24)

$$\sum_{k=0}^{n} \frac{k}{n} \binom{2n}{n-k} x^n = \binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$$

which is due to Shapiro [27].

3.2 <u>T-ary trees and generalized Catalan Numbers.</u>

We investigate the relationship of t-ary trees with generalized Catalan Numbers in the following. Knuth [11] proved that the number H_n of t-ary trees with n nodes is given by $H_n = \frac{1}{1+tn}\binom{(1+t)n}{n} = \frac{1}{(t-1)n+1}\binom{2n}{n}$ where t and n are integers.

<u>Lemma 1.</u> $\sum_{n=0}^{\infty} H_n x^n = (1+v)$ (3.2.1)

where $v = x(1+v)^p$, p is a positive integer.

<u>Proof.</u> Consider

$$\sum_{n=0}^{\infty} H_n x^n = \sum_{n=0}^{\infty} \frac{1}{1+t^n} \binom{1+t^n}{n} x^n$$

$$= \sum_{n=o}^{\infty} \frac{1}{1+tn} \frac{(1+tn)!}{n!((t-1)n+1)!} x^n$$

$$= \sum_{n=o}^{\infty} \frac{1}{n!} \frac{\Gamma(tn+1)}{\Gamma((t-1)n+2)} x^n$$

$$= \sum_{n=o}^{\infty} \frac{1}{n!} x\, T^{n-1}_{(t-1)n+2} \{1\}$$

$$= (1+v), \text{ by lemma (1.2.5) and}$$

where $v = x(1+v)^t$ and since p is also a positive integer, t can be replaced by p and the next follows.

<u>Lemma. 2</u> $$\sum_{n=o}^{\infty} C_{n+1} x^n = (\sum_{n=o}^{\infty} C_n x^n)^2 \qquad (3.2.2)$$

<u>Proof</u>. Consider

$$\sum_{n=o}^{\infty} C_{n+1} x^n = \sum_{n=o}^{\infty} \frac{1}{n+2} \frac{(2n+2)!}{(n+1)!(n+1)!} x^n$$

$$= \sum_{n=o}^{\infty} \frac{2}{n!} \frac{\Gamma(2n+2)}{\Gamma(n+3)} x^n$$

$$= \sum_{n=o}^{\infty} \frac{1}{n!} x\, T^{n-1}_{n+2+1} \{2\}$$

$$= (1+v)^2 \qquad (3.2.2)$$

where $v = x(1+v)^2$.

Lemma (1.2.5) has been used again in last step.

Consider again

$$\sum_{n=o}^{\infty} C_n x^n = \sum_{n=o}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n$$

$$= \sum_{n=o}^{\infty} \frac{1}{(n+1)} \frac{\Gamma(2n+1)}{n!\ n!} x^n$$

$$= \sum_{n=o}^{\infty} \frac{1}{n!}\, x\, T^{n-1}_{n+1+1} \{1\}$$

$$= (1+v) \qquad (3.2.3)$$

where $v = x(1+v)^2$.

We now conclude from (3.2.2) and (3.2.3)

$$\sum_{n=o}^{\infty} C_{n+1}\, x^n = \left(\sum_{n=o}^{\infty} C_n\, x^n \right)^2 \qquad (3.2.4)$$

<u>Proposition</u>. $\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(p,a)} x^{n-k} = \sum_{k=o}^{\infty} \left(\sum_{n=o}^{\infty} H_n x^n \right)^{pk+pa}$. (3.2.5)

<u>Proof</u>. In order to prove the proposition, we consider

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(p,a)}\ x^{n-k}$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} p(k+a) \frac{\Gamma(pn+pk+pa)}{n!\ \Gamma((p-1)n+pk+pa+1)} x^n$$

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{x}{n!}\, T^{n-1}_{(p-1)n+pk+pa+1} \left\{ p(k+a) \right\}$$

$$= \sum_{k=o}^{\infty} (1+v)^{pk+pa} \qquad (3.2.6)$$

where $v = x(1+v)^p$, a and p are constants.

Using lemma (3.2.1) we get from (3.2.6),

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(p,a)}\ x^{n-k} = \sum_{k=o}^{\infty} \left(\sum_{n=o}^{\infty} H_n x^n \right)^{pk+pa} \qquad (3.2.7)$$

which proves inturn the above proposition.

Remark. The result in (3.2.7) is generalization of the result (2.1.13). For p = 2, (3.2.7) yields

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(2,a)} x^{n-k} = \sum_{k=o}^{\infty} \left(\sum_{n=o}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n \right)^{2k+2a}$$

$$= \sum_{k=o}^{\infty} \left(\sum_{n=o}^{\infty} C_n x^n \right)^{2k+2a}$$

$$= \sum_{k=o}^{\infty} \left(\sum_{n=o}^{\infty} C_{n+1} x^n \right)^{k+a} \qquad (3.2.8)$$

We have used (3.2.4) in deriving (3.2.8). We conclude from (3.2.8) that

$$\sum_{n=o}^{\infty} \sum_{k=o}^{n} B_{nk}^{(2,a)} x^{n-k} = \sum_{k=o}^{\infty} \left(\sum_{n=1}^{\infty} C_n x^{n-1} \right)^{k+a}$$

which we established in Chapter II. For a = 0, this result further reduces to that of Shapiro [27].

B I B L I O G R A P H Y

1. Alter, R.

   Some remarks and results on Catalan Numbers. Proceedings Southeastern Conference on Combinatorics, Graph Theory and Computing.

2. Boas, R.P. and Buck, R.C.

   Polynomials defined by generating relations. Amer. Math. Monthly, 63, 1956, 626-632.

3. Brenke, W.C.

   On generating functions of Polynomial Systems. Amer. Math. Monthly, 52, 1945, 297-301.

4. Brown, W.G.

   Historical Note on recurrent combinatorial Problem, Amer. Math. Monthly, 72, 1965,937-977.

5. Catalan, E.

   Note sur une equation aux differences finies, J. Math. Pures Appl, (1), 3, 1836.

6. Donaghey, R.

   Restricted Plane tree Representation of Motzkin Catalan Equations, Jr. of Comb. Theory (B), 22, 114-121, 1977.

7. Finucan, H.

   Some elementary aspects of Catalan Numbers, Combinatorial Math.iv, Proceeding of Fourth Australian Conf., Aug. 27-29, 1975, 41-45.

8. Gould, H.W.

Generalization of a formula of Touchard for Catalan Numbers. Journal of Combinatorial Theory, Series A 23, 1977, 351-353.

9. Huff, W.N. and Rainville, E.D.

On Sheffer A-type of polynomial generated by $\emptyset(t)f(xt)$; Proc. Amer.Math.Soc.,3,1952,298-299

10. Izbicki, H.

Über unterbaumes eiries Baumes Monatshefte. F. Math. 74, 1970, 50-62.

11. Knuth, D.

The art of Computer Programming, Vol.I, World student Service Edition, 1975.

12. Lafer, P. and Long, C.T.

A Combinatorial Problem, The American Math. Monthly, 69, 1962, 876.

13. Laplace, M. Le Marquide.

Theorie Analytique des Probabilities; Paris 1820

14. Lois, Comtet.

Analyse combinatoire, Paris, 1970 : Tome Premier, 64 .

15. Mittal, H.B.

Some Generating Functions, Separata da revista da Faculdade de ciencias de Lisboa 2.ª Serie - A - Vol. XIII - Fasc. 1.º, 1970, 43-54.

16. Mittal, H.B.

Unusual generating relation for Polynomial sets Jr. für die reine und angewandte Mathematik, Sonderdruck aus Band 271, Seite 122 bis 138,1974.

17. Mittal, H.B.

Polynomial defined by Generating relations Transaction of the Amer. Math. Soc.,168, June 1972, 73-84.

18. Mittal, H.B.

Bilinear and Bilateral Generating Relations. Amer. Journal of Math., 99 No.1,1972,23-55.

19. Netto, E.

Lehrbuch der Combinatoric Chelsea, N.Y.1958,192.

20. Palas, F.J.

The polynomial generated by f(t) exp|p(x)u(t)|; Oklahoma Thesis, 1955.

21. Polya, G.

Mathematics and Plausible Reasoning, vol, I, Princeton Univ. Press,1954, 102 (Problems 7,8,9).

22. Rainville, E.D.

Certain Generating function and associated polynomials.Amer.Math. Monthly,52,1945,239-250.

23. Riordon, J.

A note on Catalan Paranthesis. Amer. Math. Monthly; vol. 80, 1973, 904-906.

24. Rogers, D.G.

Pascal Triangle,Catalan Triangle and Renewal Arrays, Discrete Maths, 22,1978,301-310.

25. Shapiro, L.W.

Catalan Numbers and Total Information Numbers. Proc. 6th S.E.Conf.Comb.,Graph Theory and Computing,, 1975, 531-539.

26. Shapiro, L.W.

A short proof of an identity of Touchard's concerning Catalan Numbers. Jour. of Comb. Theory, 20 No.3, May 1976.

27. Shapiro, L.W.

A Catalan Triangle,Discrete Maths.14,1976,83-90.

28. Sheffer, I.M.

Some properties of Polynomial sets of type zero. Duke Math. J. 5, 1939, 590-622.

29. Touchard, J.

Sur Certaines Equations Fonctionnelles. Proc. Int. Math. Congress,Toronto, vol.1,1924, 465-472 (1928).

30. Truesdell, C.

An essay toward a unified theory of special functions, Princeton, 1948.

31. Zeitlin, D.

On generating functions and a formula of Choudhari, Amer.Math.Month.74,1967,1056-1062.

32. Riordon, J.

Combinatorial Identities, John Wiley & Sons Inc. New York. 1968

ABSTRACT

Catalan Number has been used in finding the number of binary trees by Knuth [11].

The purpose of this thesis is to extend the standard notion of Catalan Number and to generalise some results of Shapiro [27] and Rogers [24].

The principal results are given below :

1. $B_{nk}^{(a)} = \frac{k+a}{n+a} \binom{2n+2a}{n-k}$ where a is a constant, and

i) $\sum_{k=1}^{n} B_{nk}^{(a)} = \binom{2n+2a-1}{n-1}$

ii) $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_{nk}^{(a)} x^{n-k} = \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} C_n x^{n-1} \right)^{k+a}$

iii) $B_{n+1,\, k+1}^{(a)} = B_{nk}^{(a)} + 2B_{n,\, k+1}^{(a)} + B_{n,\, k+2}^{(a)}$.

2. $B_{ns} = \frac{s}{2n-s} \binom{2n-s}{n}$

Where n and s are positive integers and $n-s \geqslant 0$ and

i) $B_{n+1,\, s+1} = B_{ns} + B_{n+1,\, s+2}$

ii) $\sum_{s=1}^{n} B_{ns} = B_{n+1,\, 2} = C_n$.

3. $B_{nk}^{(p,a)} = \frac{k+a}{n+a} \binom{pn+pa}{n-k}$

where p and a are constants and

i) $\sum_{k=0}^{n} (p-1)^k B_{nk}^{(p,a)} = \binom{pn+pa-1}{n}$

ii) $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} B_{nk}^{(p,a)} x^{n-k} = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} H_n x^n \right)^{pk+pa}$

where H_n is the number of t-ary trees with n nodes.

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