A STUDY IN EXTREME POINT MATHEMATICAL PROGRAMMING PROBLEMS

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TO

RESPECTED PARENTS

AND

ESTEEMED TEACHERS

DECLARATION

The work embodied in this dissertation has been carried out under the supervision of Dr. P.C.Saxena, School of Computor and Systems Sciences, Jawaharlal Nehru University, New Delhi - 110 067. The work is original and has not been submitted so far, in part or full, to any other University or Institution for any other degree.

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Mathematical Programming problems have well known importance in economics, Industry, Game-Theory and in the solution to many other problems of theoretical and practical importance. To study extreme point mathematical programming problems in the thesis, some of the linear and fractional functional objective functions are considered.

The total work is divided into three parts. The first part consists of four sections deals with solving the extreme point linear programming problem. In section-1 and section-2, the problem is solved by the cutting plane techniques and in section-3 and section-4, enumeration techniques are illustrated. The second part deals with an improved techniques for solving Extreme Point Linear Programming Problem. Two cuts termed as Deep Cut and strong cut are developed which are more efficient for solving extreme point linear programming problem. This also divided into two sections i.e. the cutting plane technique and enumeration technique. In the last part techniques are dealt for solving Extreme Point Linear Fractional Functional Programming Problems.

C_O_N_T_E_N_T_S

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CHAPTER - I
INTRODUCTION

This introduction traces in brief a survey of the developments in the field of mathematical programming with special emphasis on fractional programming and extreme point programming. The subject of mathematical programming has grown tremendously because of its vast applications. Here only those aspects of mathematical programming, which put the present work in its proper perspective are studied.

This introduction is divided into two sections. In the first section, the mathematical programming problem has been taken up in general. In the second section, a brief review of the related problem, such as Extreme point programming problem, Fixed charge problem Assignment problem and fractional programming problem, is given.

SECTION - I
MATHEMATICAL PROGRAMMING IN GENERAL

In the present fast changing situations, decision making authorities have to be not only objective but also very active and alert because even a slightest delay of inaccuracy can cause tremendous loses. This has led to the development of the science of operation research. Mathematical programming is one of the most important branch of operation research.

Problems which seek to maximise or minimise a numerical function of a number of variables (or functions) with the variables (or functions) subject to certain constraints, form a general class which may be called optimisation problems. The quest for solutions to those problems led to the application of differential calculus and to the development of the calculus of variations. However, many new and important optimisation problems have emerged in the field of economics, the solution of which with the help of these classical optimisation techniques tend to be tedious, long drawn and inefficient. This has led to the development of new techniques.

Broadly speaking, programming problems deal with determining optimal allocation of limited resources to meet given objectives when there are many alternatives. The general programming problem can be formulated as follows -

It is desired to determine $X = x_1 x_2 \ldots . x_3$ which satisfied the m inequalities or equations

$$g_i(x) \quad \leq, =, \geq b_i \qquad i = 1, 2, \ldots . m \quad -(1.1)$$

and in addition, maximise or minimise the function

$$Z = F(x) \quad \ldots . \quad (1.2)$$

$$\text{and } x \geq 0 \quad \ldots . \quad (1.3)$$

The restrictions (1.1) are called the constraints and (1.2) is called the objective functions and conditions (1.3) are called non-negative restrictions. In (1.1) gi(x) are assumed to be specified functions and the $b_i^{(s)}$ are assumed to be known constraints. Furthermore in (1.1) one and only one of the signs $\leq$, = $\geq$ holds for each constraint.

A special class of above mentioned programming problem is a linear programming problem where f(x) and gi(x) are all linear. In this situation the given problem can be written as

$$\text{Max (Min)} \quad Z = cx$$
$$\text{subject to} \quad Ax \{\leq, =, \geq\} b$$
$$x \geq 0$$

Where A is aij which is 'mxn' matrix. All programming problems that are not linear in the sense defined above, are called non-linear. Attention has also been paid to linear programming problems with special simple structure like transportation problems, assignment problems, network problems etc.

For a linear programming problems Dantzig's simplex method (43) is the most powerful and efficient solving technique. Simplex method solves the linear programming problem exactly in a finite number of steps or givon an indication that there is an unbounded solution.

Many practical problems, however, could not or could hardly be represented by linear programming model. Therefore, attempts have been made to develop more general mathematical programming problems. Interest in non-linear programming problems has grown simultaneously with growth of linear programming and gamo theory (43). In 1951, H.W. Kuhn and A.W.Thcker (44), published an important paper ontitled "Non linear programming" dealing with necessary and sufficient conditions for optimal solutions to programming problems which laid the foundation for great deal of later work in non-linear programming.

The mathematical programming model can be classified into four catogories -

(a) Doterministic, continuous models, the set of points satisfying all constraints, is connected and the objective function is continuous, o.g. linear programming problems, quadratic programming problems, convex programming problems, fractional functional programming problems etc.

(b) Deterministic, discontinuous models the feasible region is not connected and/or the objective function is not continuous, for example, integer linear programming, integer quadratic programming, fixed charge problem, assignment problem, extreme point fractional functional programming etc.

(c) Stochastic models: the coefficients in the constraints and/or in the objective functions are random variables. In this category we have the constrained programming problems.

(d) Dynamic Models: The coefficients, in the constraints and/or in the objective function, are dependant on a parameter say time.

The class of non-linear programming problems which has been studied most extensively, is one where the objective function is non-linear and the constraints are linear. The general problem of this kind is

$$\text{Max} \quad \text{Min} \quad Z = f(x)$$

$$\text{subject to} \quad Ax \; (\geq = \leq) \; b$$

$$x \geq 0$$

In 1954, A.Charnes and C.Lemke (9) published an approximate method of treating problems in which minimisation of separable functions subject to linear constraints when each of the separable function is convex, is studied (The function $f(x)$ is convex over a convex set 'S' in E^n if for any two points x_1 and x_2 in S and for all λ, $0 \leq \lambda \leq 1$, $f(\lambda x_1 + \overline{1-\lambda}\, x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$. Convex function is the negative of convex function. The problem of minimising a general convex function subject to linear constraints has also been considered. Most of the methods developed for this problem can be considered as large step gradient methods as given by J.B.Rosen (63) and Zoutedijk (69).

In 1955, as number of papers by different authors dealing with quadratic programming have appeared. The general quadratic programming problem is :

$$\text{Minimise} \quad Z = cx + x' Bx$$
$$\text{subject to} \quad Ax \geq b$$
$$X \geq 0$$

Where B is a semipositive definite matrix. The main contributor in this field are E.M.L.Beale (5), M.Frank and P.Wolfe (21) and P.Wolfe (67) are the well known and have an advantage of reducing a quadratic programming problem to a form which permits application of the semplex method.

An important and a Particular class of non-linear programming is convex programming in which a convex function is minimised (or a concave function is maximised) over a convex region. The well known methods for solving such a type of problems are Rosen's gradient projection method (63) Zoutendijk's method for feasible direction (69) Kelley's cutting plane method (38) and Zangwill's convex simplex method (68).

Another example of deterministic continuous model is fractional programming which deals with optimisation of the ratio of two functions subject to certain constraints. A linear fractional programming problem in its most general mathematical form is :-

$$\text{Max} \quad \frac{cx + \alpha}{Dx + B}$$

$$\text{subject to} \quad Ax (\leq = \geq) b$$

$$x \geq 0$$

Important contributors in this field are A.Charnes and W.W.Cooper (11), Bela Martos (47, 48), N.S.Dorn (17) and K.SAWRUP (33, 34).

Another class of non-linear programming problems is discrete optimisation problem where the variable are required to be non-negative integers. One of the earlier papers dealing with the subject, was published by Dantzig, Fulkerson and Johnson (15) in 1954, Gomory (26,27) was the first to set forth a systematic computational technique which converges in a finite number of iterations. Glover (24,25) and Young (64,65), Raghvachari (61) and some others have made good contributions in integer programming techniques for non-linear programming problems.

Fixed charge problem is a particular case of non-linear programming problems belonging to deterministic discontinuous category of programming models. One of the earlier papers to deal with the fixed charge problem was by Hirsch and Dantzig (31) Balinski , Juhn and Baumal and a few others concentrated on finding approximate solutions to fixed chargo transportation problems. Fixed charged problem can also be formulated as a mixed integer continuous variable linear programming problems (13).

Another deterministic models is assignment problem. It is a linear programming problem with a special structure and for its solution it is treated as zero-one integer programming problem. The linear programming formulation of this general assignment problem is

$$\text{Max} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \; x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1,2 \; \ldots\ldots \; n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1,2 \; \ldots\ldots \; n$$

$$x_{ij} = 0 \text{ or } 1$$

$$x_{ij} \geq 0 \qquad i,j = 1,2 \; \ldots\ldots \; n$$

A number of computationally efficient algorithm were developed by Balinski and Gomory (4) Khun (45) and Ford and Fulkerson (20) to solve the above problem.

Work has also done on Extreme point programming which belongs to the class of deterministic discrete models. In extreme point programming problems an objective function is optimised over a convex polyhedron with an additional requirement that optimal solution should be an extreme point of convex polyhedron. This type of problems have been studied by Kirby, M.J.L., Love, H.R. and K.Sawrup (39, 40, 41). The most general mathematical form of extreme point linear programming (E.P.L.P.) problem is

$$\text{Max} \quad Z = cx$$

$$\text{subject to} \quad Ax = b$$

$$\text{and} \quad x \quad \text{is an extreme point of}$$

$$Dx = d$$

$$x \geqslant 0$$

Alongwith the development of extreme point linear programming (E.P.L.P.) progress also been made in extreme point fractional functional programming (E.P.L.F.F.P.) which can be formulated as

$$\text{Max} \quad Z = \frac{cx + \alpha}{Dx = \beta}$$

$$\text{subject to} \quad Ax = b$$

and x is an extreme point.

$$Rx = t$$
$$x \geq 0$$

Programming problems in which some of the parameters are random variables are known as stochastic programming problems. These are almost always non-linear, a lot of work on stochastic programming has been done (5,10,14,66).

Another important technique for solving optimisation problem is Dynamic Programming. Richard Bellman and S. Dreyfus (7,8) are important contributors in the development of dynamic programming.

SECTION - 2
RELATED WORK

In this section, a brief study of the problem which related to the present work, is made. E.P.L.P. in its most general form, was first studied and solved by M.J.L. Kirby, H.R.Love and Kanti Sawrup (40). The problem can be stated analytically as

$$\left.\begin{array}{l} \text{Max } Z = Cx \\ \text{subject to } Ax = b \\ \text{and } x \text{ is an extreme point of} \\ Dx = d \\ x \geq 0 \end{array}\right\} \quad \text{Problem (2.1.1)}$$

Any zero-one integer programming problem can be converted into the above form by replacing the requirement, that each of the variables be either zero or one, by the condition that an optimal solution be an extreme point of I_n, $x \leq 1$, $x \geq 0$. The problem (2.1.1) is a larger class of problems than the class of integer programming problems.

To solve problem (2.1.1) the following linear programming problem was considered.

$$\text{Max } Z = Cx$$
$$\text{subject to } Fx = f$$
$$x \geq 0$$

where $F = \begin{bmatrix} A \\ D \end{bmatrix}$ and $f = \begin{bmatrix} b \\ d \end{bmatrix}$

optimal extreme point solutions (which hereinafter have also been termed as first best extreme point solutions, second best extreme point solutions were obtained and 3rd best, 4th best extreme point solutions of problem (2.1.2) were determined by a cutting plane method. These points were tested at each stage whether an extreme point of (2.1.2) obtained at that stage, is also an extreme point of $Dx = d$, $x \geq 0$ the ith best extreme point solutions of the problem (2.1.2) are second best extreme point solutions of the problem :

$$\text{Max } Z = Cx$$
$$\text{subject to } Fx = f$$
$$Cx \leq U'_{i-1}$$
$$x \geq 0$$

where $U'_{(i-1)}$ is the value of the objective function at $(i-1)^{th}$ best extreme point solutions of (2.1.2). A method of finding second best extreme point solution was developed in (40). At some stage, when an extreme point solution of (2.1.2) was reached which was also an extreme point of $Dx = d$, $x \geq 0$, the process was terminated. If a stage was reached, when no further best extreme point solution could be possible the problem (2.1.1) was said to have no solution. In this process, besides the computational difficulty of testing whether an extreme point of $Dx = d$, $x \geq 0$, another difficulty of finding alternative optima at each step was faced.

The difficulty of testing at each stage whether an extreme point of (2.1.2) is an extreme point of $Dx=d$, $x \geq 0$ was avoided in (41) where the extreme solution of the following problem -

$$\left.\begin{array}{l} \text{Max} \quad Z = Cx \\ \text{subject to} \quad Dx = d \\ \qquad\qquad\quad x \geq 0 \end{array}\right\} \text{Problem (2.1.1)}$$

were determined in a systematic order starting from first best extreme point solutions by the cutting plane method till either feasibility in $Ax = b$ was achieved or some indication of no solution was obtained. In this approach alternative optima were still needed at each stage.

In (39), an enumerative technique was developed in which the difficulty of obtaining alternative optima at each stage was also get rid off. In this approach extreme point solution of (2.2.1) were enumerated in a systematic order till either feasibility in $Ax = b$ was achieved or some indication of no solution was obtained.

Fixed charge problem (F.C.P.) is a particular form of a non-linear programming problem. In its most general form it is -

$$\text{Min } Z = \sum_{j=1}^{n} \phi_j(x_j)$$

$$\text{subject to } Ax = b$$

$$x \geq 0$$

$$\delta_j = 0 \text{ if } x_j = 0$$

$$= 1 \text{ if } x_j > 0$$

where $\phi_j(x_j) = c_j x_j + f_j \delta_j$, $f_j > 0$ $j = 1,2 \ldots\ldots n$

The number fj's are called fixed charges, since fj is incurred only if $xj > 0$. If it were not for fixed charges, F.C.P. will be a simple linear programming problem. As the function $\phi_j x_j$, $j = 1, 1, \ldots\ldots n$ are concave for $x_j \geq 0$, $j = 1,2,\ldots\ldots n$ and the objective function is being minimised optimal solution of the problem lies at an extreme point of the convex set of feasible solutions. However, there can be local optima different from global optima. Approximate methods were developed which could determine only local optima for a F.C.P. Hirsch and Dantzig (31) made a simple observation that optimal solution must occur at an extreme point of convex set of feasible solutions. The method of ranking the extreme point of $Ax = b$, $x \geqslant 0$, as developed by Murty (49), to obtain an exact solution of F.C.P., is useful when the fixed charges are quite small compared to the range in the values of the variable costs. Murty (50) and Gray (28) developed approaches for solving a fixed charge transportation problem. Murty's approach is useful where fixed charges are quite small compared to the transportation cost whereas Gray's approach is useful when fixed charges dominate and have an upper bound.

In (13), F.C.P. is converted into a mixed integer continuous variables linear programming problem of the form

$$\text{Min } Z = \sum_{j=1}^{n} c_j x_j + f_j \delta_j$$

subject to $Ax = b$

$$x_j - d_j \delta_j \quad 0 \qquad j = 1, 2, \ldots . n$$

$$0 \leq d_j \leq 1 \qquad j = 1, 2, \ldots . n$$

δ_j integers

$x \geq 0$

where d_j's are upper bounds of x_j's in $Ax = b$ assignment problem, in general, deals with assignment 'n' persons to 'n' jobs in such a way that the total value to the company is maximised. If x_{ij} is the value of the assignment of i^{th} person to j^{th} job and further c_{ij} be the value of i^{th} person to the company if he is assigned to j^{th} job, then the most general form of assignment problem (23) is

$$\text{Max} \quad Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \; x_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, \ldots\ldots \; n$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2, \ldots\ldots \; n$$

$$x_{ij} \geqslant 0$$

As this is a particular form of transportation problem it can be solved by the same techniques as are used to solve a transportation problem. Both Dwyer (18) and Votaw and Orden have discussed methods for obtaining 'near -optimal' solutions to the assignment problem and, of course, to the transportation problem. The special and simple structure of the constraints (viz. $\sum_i x_{kj} = 1$ $\sum_j x_{ij} = 1$) has led to the development of a number of computationally efficient algorithms (4,12,20,45).

Additional techniques have also been developed by Von Neumann (51) and Egervary (19).

Linear Fractional Functional Programming (L.F.F.P.) is a special class of non-linear programming. This special class of programming problems are concerned with optimising a given ratio of two function of non-negative variables subject to certain linear and/or non-linear constraints. These problems are distinct from convex/concave programming problems because the objective function to be optimised is neither convex nor concave mathematical model for a general linear fractional programming problem is

$$\text{Max} \quad Z = \frac{Cx + \alpha}{Dx + \beta}$$

$$\text{subject to} \quad Ax \leq b$$

$$x \geq 0$$

Charnes, A. and Cooper W.W. (11) in 1962 replaced any L.F.F.P. problem by at the most two ordinary linear programming problems. The equivalent linear programms of the above model are

$$\text{Max} \quad Cy = \alpha t$$

$$\text{subject to} \begin{cases} Ay - bt \leq 0 \\ Dy + \beta t = 1 \end{cases}$$

$$Y, t \geq 0$$

$$\begin{aligned} \text{Max} \quad & - cy - \alpha t \\ \text{subject to} \quad & Ay - bt \leq 0 \\ & -Dy - \beta t = 1 \\ & y, t \geq 0 \end{aligned}$$

If the sign of the denominator is known, only one of these problems is required to be solved.

In 1964, Bela Martos (47) developed a computational simplex technique for L.F.F.P. under the title "Hyperbalic Programming" in two cases called simple case (where solution set is regular and denominator at the objective function for all feasible solutions is strictly positive) and general case.

In simple case, it is shown that the objective function has a finite maximum which is achieved on, at least, one of the vertices of the convex set 'S' of feasible solutions and computational technique is described. In the general case, when the conditions for the simple case are absent, the problem may have an optimal solution even through the set 'S' is unbounded and the denominator becomes zero.

In 1965, Bela Martos (48) proved that the linear fractional function (which is neither convex nor concave) is quasi monotonic. Because of the quast monotonicity of this function, it has the following two important properties -

1. A local maximum is a global maximum
2. The maximum of it occurs at an extreme point of the set 'S'.

In (33), K.SAWRUP has given simplex-like iterative procedure for the solution of linear fractional programmes. The problem considered is

$$\text{Max} \quad Z = \frac{Cx + \alpha}{Dx + \beta}$$

$$\text{subject to} \quad Ax = b$$

$$x \geqslant 0$$

where the constraint set is regular and the denominator $Dx+\beta$ is positive over the constraint set. K.Sawrup solved this problem directly beginning with a basic feasible solution and the conditions under which the solution could be improved were obtained. The optimality conditions

$\Delta_j \geqslant 0$ for all 'j' were also established where

$$\Delta_j = Z^2 (Z_j' - C_j) - Z' (Z_j^2 - d_j)$$

In (34), K.Sawrup has developed an algorithm on the basis similar to that adopted by E.M.L. Beale (6) for the solution of quadratic programming and the basic result is that for the maximum at a basic feasible solution

$$\frac{\delta z}{\delta z_j} \leq 0 \qquad j = 1,2 \ldots . \; n$$

$$\text{where } Z = \frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j}$$

$$\text{and } X_s = \gamma_{s0} + \sum_{q=1}^{n} \gamma_{sq} Z_q \text{ , } (s = 1, 2 \ldots\ldots n)$$

(Z_j's' are non-basic variables and x_s's' are basic variables)

Certain relation and common characteristics of linear fractional program and its equivalent linear programs are also established. Dual simplex type algorith is developed for the linear fractional functional programming problem. Finally, outlines of a technique for obtaining an integer solution to the linear fractional functional programming problem based on the integer linear programming algorithm of Gomory (26) has been discussed. Simultaneously with the fast development of research in linear fractional programming, the field of non-linear fractional programming has also developed. Many papers, such as, Almogy and Levin (1) Dinkelback(16), Jagannathan (32), Mangasarian (46), K.Sawrup (35) etc. have appeared on the subject :

Some work has been done on extreme point linear fractional functional programming by K.Sawrup and R.K.Gupta (36, 37). The problem considered by them is

$$\text{Max } Z = \frac{Cx + \alpha}{Dx + \beta}$$

subject to $Ax = b$

and x is an extreme point of

$Rx = t$

$x \geq 0$

In (36) the extreme point of $Ax = b$, $Rx = t$ $x \geq o$ are determined in a systematic order by a cutting plane method till extreme point $Rx = t$, $x \geq o$ is reached. In this approach, at each stage difficulties of finding alternative optima and testing whether an extreme point of $Rx = t$, $x \geq o$ are faced. In (37) the latter difficulty is got rid off by determining the extreme points of $Rx = t$, $x \geq o$ in a systematic order by a cutting plane method. The process is terminated when feasibility in $Ax = b$ is achieved.

CHAPTER – II
EXTREME POINT LINEAR PROGRAMMING PROBLEM

This chapter, consisting of four sections, deals with solving the Extreme Point Linear Programming Problem. Section I and Section 2 solve the problem by cutting plane techniques while in Section 3 and Section 4 enumeration techniques are presented for solving the problem.

SECTION – I
CUTTING PLANE PROCEDURE I

INTRODUCTION :

The Extreme Point Linear Programming Problem (EPLPP) in its most general form was first studied and solved by M.J.L Kirby, H.R.Love and Kanti Swarup. EPLPP in general, is stated as

$$\left.\begin{array}{ll} \text{Max} & Z = Cx \\ \text{subject to} & Ax = b \\ & X \geq 0 \\ \text{and } X & \text{is an extreme point of} \\ & Dx = d \\ & X \geq 0 \end{array}\right\} \quad (II.1.1)$$

where A is $(m \times n)$ matrix, D is $(p \times n)$ matrix, b is $(m \times 1)$ matrix, d is $(p \times 1)$ matrix and X is $(n \times 1)$ matrix. The procedure presented in this section solves (II.1.1) by cutting plane technique where one moves from one extreme point to another extreme point of the convex polyhydron $Ax = b$ $Dx = d$, $X \geq 0$. till an extreme point of $Dx = d$, $X \geq 0$ is reached.

THEORETICAL DEVELOPMENT :

To solve the problem (II.1.1) consider the linear programming problem (LPP)

$$\begin{array}{rl} \text{Max} & Z = Cx \\ \text{subject to} & \left.\begin{array}{l} Ax = b \\ Dx = d \\ X \geq o \end{array}\right\} \equiv FX = f \end{array} \qquad \text{(II.1.2)}$$

Here F becomes a (m + p) Xn matrix and F, a (m + p) X 1 matrix

It may be noted that problem (II.1.1) is always bounded because any solution of (II.1.1) is an extreme point of $Dx = d$, $x \geq o$ and this set of extreme points is always finite. But the problem (II.1.2) may be bounded or unbounded. In case problem (II.1.2) is unbounded, it can always be converted into a bounded problem introducing an additional constraint $CX \leq M$, where M is an arbitrary large, finite positive number determined in such a way that none of the extreme points of (II.1.1) are excluded with the inclusion of this additional constraint.

Thus the problem (II.1.2) henceforth is always considered to be bounded.

A few notations are introduced to develop the theory.

NOTATIONS :-

$J = [d_j : d_j \neq 0$, where d_j is j^{th} column of $D]$

$J(X) = [d_j \in J : x_j \neq 0$ where $X = (x_1, x_2, \ldots x_n)]$

$S_1 = [X : AX = b$ and X is extreme point of $DX = d, X \geq 0]$

$S_2 = [X :$ is an extreme point of $FX = f, X \geq 0]$

$S_3 = S_2 - S_1$

$X_1^{(2)} = X_{11}^{(2)}, X_{12}^{(2)} \ldots\ldots X_{1h_1}^{(2)}$ is the set of all optimal extreme point solutions of the problem (II.1.2)

$V_1^{(2)} = CX_{11}^{(2)}$, an element of $X_1^{(2)}$

THEOREM 1 : Every extreme point of $DX = d, X \geq 0$ satisfying feasibility in $AX = b$ is also an extreme point of $FX = f, X \geq 0$ i.e. $S_1 \subseteq S_2$

PROOF : If $S_1 = \emptyset$, theorem is done.

If $S_1 \neq \emptyset$, let $X \in S_1$. Thus X is an extreme point of $Dx = d, X \geq 0$ and satisfies $AX = b$

$\therefore$ X satisfies $\begin{pmatrix} A \\ D \end{pmatrix} X = \begin{pmatrix} b \\ d \end{pmatrix}$

i.e. X satisfies $FX = f$

Thus X is an element of the convex set $\bar{S}_2 = (X : FX=f, X \geq 0)$

It is required to show that X is an extreme point of $\bar{S}_2$ i.e. $X \in S_2$.

If $X \notin S_2$, $\exists t, w \in \bar{S}_2$, $t \neq w$ such that $X = \lambda t + (1 - \lambda) w$, $0 < \lambda < 1$.

As $t, w \in \bar{S}_2$, therefore $Ft = f$, $Fw = f$; $t, w \geq 0$

$Ft = f \implies \begin{pmatrix} A \\ D \end{pmatrix} t = \begin{pmatrix} b \\ d \end{pmatrix}$

$\implies Dt = d$

$Dw = d$

Similarly

Thus t, w are distinct points of the set

$(X : DX = d, X \geq 0)$

such that X is their linear convex combination.

Therefore, this X is not extreme point of $DX = d$, $X \geq 0$.

That is $X \notin S_1$; a contradiction

$\therefore$ X is an extreme point of $\bar{S}_2$

i.e. $X \in S_2$

Since X is any element of S_1, therefore $S_1 \subseteq S_2$.

Hence the result

This theorem guarentees that an optimal solution to the problem (II.1.1) is an extreme point of (II.1.2). One can also find the following relations between the problems (II.1.1) and (II.1.2):

(L 1) Problem (II.1.2) and (II.1.1) both have solutions.

(L 2) Problem (II.1.2) has no solution and hence (II.1.1) has no solution.

(L 3) Problem (II.1.2) has a solution but (II.1.1) has no solution.

Apply simplex method to solve problem (II.1.2). If there is no solution to this problem then by (L2) (II.1.1) has no solution. Therefore the problem (II.1.1) has its importance only when (II.1.2) has a solution i.e. $S_2 \neq \emptyset$. Assume that both the problems have solutions

<u>THEOREM 2</u> : If $X^* \in X_1^{(2)}$, the set of all optimal extreme point solutions of (II.1.2) then an optimal solution X of (II.1.1) is an element of S_1 which is at a minimum orthogonal distance from the hyperplane

$CX = CX^* = V_1^{(2)}$, $V_1^{(2)}$ being the optimal value of the objective function of (II.1.2)

PROOF : The problem (II.1.1) can be restatedas

$$\text{Max} \quad Z = CX$$

subject to

$$X \in S_1$$

This problem is equivalent to

$$\text{Min} \quad Z = -CX$$

subject to

$$X \in S_1$$

which is further equivalent to

$$\text{Min} \quad \frac{V_1^{(2)} - CX}{\|C\|}$$

subject to

$$X \in S_1$$

$V_1^{(2)}$ and $\|C\|$ being constants.

Let X be the optimal solution of (II.1.1) then $X \in S_1$. Since $S_1 \subseteq S_2$, therefore $X \in S_2$. Also since X^* is the optimal solution of the problem (II.1.2), therefore $CX^* \geq CX \;\forall\; X \in S_2$

$$\therefore \quad V_1^{(2)} \geq CX$$

$$\text{or} \quad V_1^{(2)} - CX \geq 0$$

$$\text{or} \quad \frac{V_1^{(2)} - CX}{} \geq 0$$

$\frac{V_1^{(2)} - CX}{C}$ represents the orthogonal distance of X from hyperplane $CX = CX^* = V_1^{(2)}$.

Since tho problem (II.1.1) as restated in the above form seeks to minimize this orthogonal distance subject to $X \in S_1$, thus X, the optimal extreme point solution of (II.1.1) is at a minimum orthogonal distance from the hyperplane $CX = V_1^{(2)}$.

Thus the problem reduces of finding an $X \in S_1$ which is at a minimum orthogonal distance from the hyperplane $CX = V_1^{(2)}$. To achieve this, the concept of 2nd best, 3rd best etc. extreme point solutions are introduced.

<u>Second Best Extreme Point Solution</u> : The second best extreme point solution to a LPP (II.1.2) is an element $s^* \in (S_2 - X_1^{(2)})$ such that $cs^* \geq cs \;\forall\; s \in (S_2 - X_1^{(2)})$, where $X_1^{(2)}$ is the set of optimal extreme point solutions of (II.1.2). Let $X_2^{(2)} = (X_{21}^{(2)}, X_{22}^{(2)} \ldots\ldots\ldots X_{2h_2}^{(2)})$ be the set of all second best extreme point solutions of (II.1.2).

Similarly, the third best extreme point solution to problem (II.1.2) is an element $s^* \in (S_2 - (X_1^{(2)} \cup X_2^{(2)})$ such that $cs^* \geq cs \;\forall\; s \in (S_2 - (X_1^{(2)} \cup X_2^{(2)}))$. The set of third best extreme point solutions of (II.1.2) is denoted by $X_3^{(2)} = \{X_{31}^{(2)}, X_{32}^{(2)} \ldots\ldots X_{3h_3}^{(2)}\}$

In general the N^{th} best extreme point solution of (II.1.2) is defined an element $s^* \in (S_2 - (\bigcup_{i=1}^{N-1} X_i^{(2)}))$ such that $cs^* \geq cs \;\forall\; s \in (S_2 - (\bigcup_{i=1}^{N-1} X_i^{(2)}))$,

$X_i^{(2)}$, $i = 1,2, \ldots N-1$ i^{th} best extreme point solution of (II.1.2)

LEMMA1 : If a LPP has a second best extreme point solution then it is adjacent to some optimal extreme point.

PROOF : Consider the simplex table corresponding to an element of the set of second best extreme point solution of (II.1.2). Since it is not optimal there must exist at least one column, say j^{th}, for which $z_j - c_j$ o. If the corresponding column a_j is entered in the basis departing a column for which $\theta = \underset{i}{\text{Min}} \left\{ \frac{X_{Bi}}{Y_{ij}} ; Y_{ij} > 0 \right\} > 0$ then this single simplex iteration will lead to optimal solution of LPP. As only a change of one basis vector in the second best leads to optimal solution, it follows that the second best is adjacent to some element of optimal extreme point solution.

<u>LEMMA 2</u> : If a LPP has h^{th} best extreme point solution then it is adjacent to some element of the set containing optimal, 2nd best, ... (h - 1) st best extreme point solution.

<u>PROOF</u> : Consider the simplex table corresponding to h^{th} best extreme point solution of (II.1.2), $h > I$. Since it is not the optimal solution, there must exist at least one column, say h^{th}, for which $Z_j - c_j < o$. If the j^{th} column, with corresponding $z_j - c_j < o$, enters the basis departing a column corresponding to $\theta = \text{Min} \left[\frac{x_{Bi}}{yi_j}, yi_j > o\right]$, $\theta > o$, then the simplex table so obtained will generate an element of $\bigcup_{i=1}^{h-1} X_i^{(2)}$. As simplex method moves from one extreme point solution to another along an edge, therefore, the h^{th} best extreme point solution to a LPP is adjacent to an element of $\bigcup_{i=1}^{h-1} X_i^{(2)}$.

<u>LEMMA 3</u> : If $X^* \in X_1^{(2)}$ is an optimal solution of (II.1.2) then a second best extreme point solution of (II.1.2) is an element of $S_2 - (X_1^{(2)})$ which is at a minimum orthogonal distance from the hyperplane $CX = CX^* = V_1^{(2)}$.

<u>PROOF</u> : Since X^* is an optimal solution to (II.1.2), therefore, $CX^* \geq CX \;\forall\; X \in S_2$. Let $Y^* \in S_2 - X_1^{(2)}$ be the second best extreme point solution to (II.1.2).

$$\therefore \; CY^{*} \geq VY \;\; \forall \; Y \in S_2 - X_1^{(2)}$$

$$\text{i.e.} \; -CY^{*} \leq -CY \;\; \forall \; Y \in S_2 - X_1^{(2)}$$

$$CX^{*} - CY^{*} \leq CX^{*} - CY \;\; \forall \; Y \in S_2 - X_1^{(2)}$$

$$\frac{CX^{*} - CY}{\| c \|} \leq \frac{CX^{*} - CY}{\| c \|} \;\; \forall \; Y \in S_2 \; X_1^{(2)}$$

$$\frac{V_1^{(2)} - CY^{*}}{\| c \|} \leq \frac{V_1^{(2)} - CY}{\| c \|} \;\; \forall \; Y \in S_2 - X_1^{(2)}$$

Thus the orthogonal distance of Y^{*} from the hyperplane $CX = V_1^{(2)}$ is less than or equal to distance of any other point $Y \in S_2 - X_1^{(2)}$ from the same hyperplane. Hence Y^{*} is at a minimum orthogonal distance.

PROCEDURE :

Let the rank of D be p and the rank of F be $(m + p)$. If $X \in S_2$ then it has at most $(M + p)$ non-zero components. Also any extreme point solution of (II.1.1) has at most p non-zero components. Thus if $X \in S_1 \cap S_2$ then $J(X)$ has at the most p non-zero components. Thus if $X \in S_2$ and $|J(X)| > p$ then $C \notin S_1$ and if $X \in S_2$ and $|J(X)| \leq p$ then $X \in S_1$ if elements of $J(X)$ are linearly independant.

Thus an extreme point solution of (II.1.2) is also an extreme point solution of (II.1.2) $|J(X)| \leq p$ and elements of $J(X)$ are linearly independent.

Solve the problem (II.1.2) by simplex method. This yields $X_1^{(2)}$, the set of optimal extreme point solutions of (II.1.2) with $V_1^{(2)}$ as the value of objective function. If $X_1^{(2)} \cap S_1 \neq \emptyset$ then every element of $X_1^{(2)} \cap S_1$ is an optimal solution of (II.1.1) and the process terminates. In case $X_1^{(2)} \cap S_1 = \emptyset$, then by Theorem 1 an optimal solution of (II.1.1) will be an element of $S_1 \cap (S_2 - X_1^{(2)})$. Determine the set $X_2^{(2)} = \{X_{21}^{(2)}, X_{22}^{(2)} \ldots\ldots X_{2h_2}^{(2)}\}$ of all second best extreme point solution of (II.1.2) by using procedure T as detailed below.

Procedure T :

Let B be the basis corresponding to an element of $X_1^{(2)}$, X_B be the vector of the basic variables and C_B be the row vector with components as the coefficient associated with the basic variables in the objective function.

Thus $X_B = B^{-1} f_j$

$$Y_j = B^{-1} f_j$$

$$Z_j = C_B Y_j$$

For each element of $X_1^{(2)}$ determine

$$H(B) = \{ j : z_j - c_j > 0 \}$$

$$Q_j = \underset{i}{\mathrm{Min}} \left\{ \frac{X_{Bi}}{Y_{ij}} , \; y_{ij} > 0 \right\} , \; j \in H\,(B)$$

$$Y_B = \left\{ \underset{j \in H(B)}{\mathrm{Min}} \quad Q_j(Z_j - C_j), \; Q_j > 0 \right\}$$

and $S = \mathrm{Min} \; \{ Y_B : B$ is a basis for an element of X_1^{kh}

This process determines the optimal table to be used and the column to enter and leave the basis to determine second best extreme point solution of (II.1.1). If various minima obtained in T are unique, the second best extreme point solution is unique, otherwise, the set of second best extreme point solutions is generated. Let $X_2^{(2)} = X_{21}^{(2)}, X_{22}^{(2)}, \ldots\ldots X_{2h_2}^{(2)}$ be the set of second best extreme point solutions of (II.1.2) and $V_2^{(2)}$ be the value of the objective function corresponding to an element of $X_2^{(2)}$. Determine $X_2^{(2)} \cap S_1$. In case $X_2^{(2)} \cap S_1 \neq \emptyset$ then every element of $X_2^{(2)} \cap S_1$ is an optimal solution of (II.1.1), otherwise, the set of third best extreme point solutions to problem (II.1.1) is determined by introducing an additional constraint $CX \quad V_2^{(2)}$ termed as a 'cut'.

This gives rise to a new problem

$$\left.\begin{array}{l} \text{Max} \quad Z = CX \\ \text{subject to} \\ \quad FX = f \\ \quad CX \leq V_2^{(2)} \\ \quad X \geq 0 \end{array}\right\} \quad \text{(II.1.3)}$$

Solve the problem (II.1.3). Let $X_1^{(3)}$ be the set of optimal extreme point solutions of (II.1.3). Clearly $X_2^{(2)} \subseteq X_1^{(3)}$. Any second best extreme point solution of $\{$(II.1.3) is an element $Y^* \in P$ where

$$P = \left[\{ X : X \text{ is an extreme point of } FX = f \; cx \leq v_2^{(2)}, \; x \geq 0 \} - x_1^{(3)} \right]$$

such that $CY^* \geq CY \;\forall\; Y \in P$. But as $CX = V_2^{(2)}$ for all $X \in X_1^{(3)}$ thus the second best extreme point solution of (II.1.3) is an element Y^* of the set $\{ X : X$ is extreme point of $FX = f$, $cx \;\; v_2^{(2)}$, $x \geq 0 \}$
$\equiv S_2 - (X_1^{(2)} \cup X_2^{(2)}$

such that $CY^* \geq CY \;\forall\; Y \in S_2 - (X_1^{(2)} \cup X_2^{(2)})$.

Thus the second best extreme point solutions of (II.1.3) is third best extreme point solution of (II.1.2) i.e. $X_2^{(3)} \equiv X_3^{(2)}$. let $X_3^{(2)} = \{ X_{31}^{(2)}, X_{32}^{(2)}, \ldots . X_{3h_3}^{(2)} \}$ be the set of third best extreme point solutions of (II.1.2) and $V_3^{(2)}$ be the value of the objective function corresponding to any element of $X_3^{(2)}$.

Determine $X_3^{(2)} \cap S_1$. If $X_3^{(2)} \cap S_1 \neq \emptyset$ then every element of $X_3^{(2)} \cap S_1$ is optimal solution for (II.1.1), otherwise, find the next best extreme point solution of (II.1.1) by introducing the another cut $CX \; V_3^{(2)}$ in (II.1.2) and determine the second best extreme point solution of the problem

$$\left.\begin{aligned} &\text{Max} \quad Z = CX \\ &\text{subject to} \\ &\qquad FX = f \\ &\qquad CX \leq V_3^{(2)} \\ &\qquad X \geq 0 \end{aligned}\right\} \quad \text{(II.1.4)}$$

which will generate the set $X_4^{(2)}$ of fourth best extreme point solution of (II.1.2). Again test if $X_4^{(2)} \cap S_1 \neq \emptyset$. This process of introducing cuts in order to obtain the next best extreme point solution of (II.1.2) is continued till at some stage, say h^{th}, the set of h^{th} best extreme point solutions of (II.1.2) is such that $X_h^{(2)} \cap S_1 \neq \emptyset$, where $X_h^{(2)}$ is the second best extreme point solutions of the problem.

$$\left.\begin{aligned} &\text{Max} \quad Z = CX \\ &\text{subject to} \\ &\qquad FX = f \\ &\qquad CX \leq V_{h-1}^{(2)} \\ &\qquad C \geq 0 \end{aligned}\right\} \quad \text{(II.1.h)}$$

$V_{h-1}^{(2)}$ is the value of objective function at an element of (h-1)st best extreme point solution.

NOTE : The cutting planes $CX \leq V_i^{(2)}$, i 1 1,2 ... h-1 are parallel to each other and $V_{i-1}^{(2)} \geq V_i^{(2)}$ $\forall$ i the constraint $CX \leq V_i^{(2)}$ makes the constraints $CX \leq V_j^{(2)}$ j = 1,2 .... i - 1 redundent. Hence only one constraint (cutting plane) is considered at any stage.

The procedure as discussed above converges because :

(a) $S_1 \subseteq S_2$ and S_2 is finite

(b) No set $X_i^{(2)}$ is repeated as $V_i^{(2)} > V_{i+1}^{(2)}$ $\forall$ i

<u>THEOREM 3</u> : If $\hat{x}$ is an optimal solution of (II.1.1) and $X_1^{(2)} \cap S_1 = \emptyset$ then $\hat{X}$ is adjacent to some element of $S_3 = S_2 - S_1$. Moreover, for all points adjacent to some element of S_2, $\hat{X}$ is at a minimum orthogonal distance from the hyperplane $CX = V_1^{(2)}$.

<u>PROOF 1</u> : $\hat{X}$ is optimal solution of (II.1.1), therefore, $\hat{X} \in S_1$. By theorem 2, $\hat{X}$ is at a minimum orthogonal distance from the hyperplane $CX = V_1^{(2)}$.

Now $\hat{X} \in S_1$; $S_1 \subseteq S_2$ therefore $\hat{X} \in S_2$. Let $\hat{X}$

be N^{th} best extreme point solution of (II.1.2), therefore $X_i^{(2)} \cap S_1 = \emptyset$, $i = 1, \ldots\ldots (N - 1)$ and $\bigcup_{i=1}^{N-1} X_i^{(2)} \subseteq S_3$. Also, since X is N^{th} best extreme point solution of (II.1.2), bu Lemma 2, it is adjacent to some element of S_3 .

It may thus be noted in the above procedure all the extreme points of S_2 are not required to be examined , the procedure begins with the best extreme points of problem (II.1.2), and proceeds to study in order, the second best, third best, ... N^{th} best extreme points and terminates as soon as an extreme point of (II.1.1) is arrived at.

In case problem (II.1.2) has a solution but problem (II.1.1) has no solution, all the extreme points of (II.1.2) are tested. Since (II.1.1) has not solution, therefore, $S_1 = \emptyset$ and $X_i^{(2)} \cap S_1 = \emptyset \; \forall \; i$, suggesting procedure will continue indefinitly. But this is impossible as S_2 is finite and $V_i^{(2)} \geq V_{i+1}^{(2)}$ $\forall i$ and so after a finite number of steps say $\bar{N}$ it will be impossible to find out second best extreme point solution to the problem.

$$\text{Max } Z = CX$$

$$\text{subject to}$$

$$FX = f$$

$$CX \leq V_N^{(2)}$$

$$X \geq 0$$

which is indicated by the fact that there is no optimal simplex table with a j such that $Q_j > 0$ and $Z_j - C_j > 0$

It may also be observed that if the problem (II.1.2) is unbounded, it is converted into a bounded problem as discussed earlier. In this case the optimal solution $X_o^{(2)}$ of the converted problem may not be an extreme point of the convex polyhydron $FX = f, X \geq o$ and in this case $X_1^{(2)}$, the second best extreme point solution of the converted problem, will be an extreme point of $FX = f, X \geq 0$. Taking $X_1^{(2)}$ as the starting solution the procedure follows exactly for the case when (II.1.2) is bounded.

EXAMPLE :

$$\text{Max } Z = 2X_1 + 3X_2$$

subject to

$$-x_1 + 2x_2 \leq 8$$

$$x_1 + x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

and $-(x_1, x_2)$ is extreme point of

$$-x_1 + x_2 \leq 3$$

$$x_2 \leq 6$$

$$2x_1 - x_2 \leq 12$$

$$3x_1 + x_2 \leq 27$$

$$x_1, x_2 \geq 0$$

SOLUTION :

For solving N(II.1.1) start with the problem :

$$\left.\begin{array}{l} \text{Max } Z = 2x_1 + 3x_2 \\ \text{subject to} \\ FX = f \quad \left\{\begin{array}{l} -x_1 + 2x_2 + x_3 = 8 \\ x_1 + x_2 + x_4 = 12 \\ -x_1 + x_2 + x_5 = 3 \\ x_2 + x_6 = 6 \\ 2x_1 - x_2 + x_7 = 12 \\ 3x_1 + x_2 + x_8 = 27 \end{array}\right. \\ X \geqslant 0 \quad x_1, x_2, \ldots x_8 \geqslant 0 \end{array}\right| \quad N\ (II.1.2)$$

The set $X_1^{(2)}$ of optimal solutions of N (II.1.2) is

$$X_1^{(2)} = \{X_{11}^{(2)} = (6,6,2,0,3,0,6,3)\}$$

and $V_1^{(2)} = 30$

Now $J = \{d_1, d_2, d_5, d_6, d_7, d_8\}$

and p, Rank of D = 4

$$J\ (X_{11}^{(2)}) = \{d_1, d_2, d_5, d_7, d_8\}$$

$$|J\ (X_{11}^{(2)})| = 5 > 4$$

$$X_1^{(2)} \cap S_1 = \emptyset$$

Proceed to find out $X_2^{(2)}$, the set of second best extreme point solutions of N(II.1.2) as follows:

$$H(B) = (4,6)$$

$$Q_4 = \text{Min } (6,2,3) = 2$$

$$Q_5 = \text{Min } (6,2,3/2) = 3/2$$

$$\gamma_B = \text{Min } (2x2,\ 3/2x1) = 3/2$$

$$\delta = 3/2$$

Therefore f_6 enters and f_8 leaves the basis to give a second best extreme point solution of N(II.1.2).

$$X_2^{(2)} = X_{21}^{(2)} = \left(\frac{15}{2}, \frac{9}{2}, \frac{13}{2}, 0, 6, \frac{3}{2}, \frac{3}{2}, 0 \right)$$

$$V_2^{(2)} = \frac{57}{2}$$

Simplex Table For $X_{21}^{(2)}$

	Variables of	C_j	2	3	0	0	0	0	0	0
C_B	Basis	X_B	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
2	f_1	15/2	1	0	0	-1/2	0	0	0	1/2
0	f_3	13/2	0	0	1	-7/2	0	0	0	3/2
3	f_2	9/2	0	1	0	3/2	0	0	0	-1/2
0	f_5	6	0	0	0	-2	1	0	0	1
0	f_7	3/2	0	0	0	5/2	0	0	1	-3/2
0	f_6	3/2	0	0	0	-3/2	0	1	0	1/2
	z=57/2	z_j-c_j	0	0	0	7/2	0	0	0	-1/2

$$|J(X_{21}^{(2)})| = d_1, d_2, d_5, d_6, d_7$$

$$J(X_{21}^{(2)}) = 5 > 4$$

$$X_2^{(2)} \cap S_1 = \emptyset$$

Introduce the cut $2x_1 + 3x_2 \leq \frac{57}{2}$ in N (II.1.2)

The problem so obtained is :

$$\left.\begin{array}{l} \text{Max } Z = 2x_1 + 3x_2 \\ \text{subject to} \\ -x_1 + 2x_2 + x_3 = 8 \\ x_1 + x_2 + x_4 = 12 \\ -x_1 + x_2 + x_5 = 3 \\ x_2 + x_6 = 6 \\ 2x_1 - x_2 + x_7 = 12 \\ 3x_1 + x_2 + x_8 = 27 \\ 2x_1 + 3x_2 + x_9 = 57/2 \\ x_1, x_2, \ldots\ldots, x_9 \geq 0 \end{array}\right| \quad N\ (II.1.3)$$

The set $X_1^{(3)}$ of optimal solutions of problem N(II.1.3) is

$$X_1^{(3)} = \left[\begin{array}{l} X_{11}^{(3)} = (\frac{21}{4}, 6\frac{5}{4}, 0\ \frac{9}{4}, 0\ \frac{15}{2}, \frac{21}{4}, 0) \\ X_{12}^{(3)} = (\frac{15}{2}, \frac{9}{2}, \frac{13}{2}, 0, 6, \frac{3}{2}, 0, 0) \end{array} \right]$$

Find out the set $X_2^{(3)}$ of second best extreme point solutions of N (II.1.3)

Let B_1 be basis for $X_{11}^{(3)}$ and B_2 for $X_{12}^{(3)}$.

For B_1

$$H(B_1) = [9]$$

$$Q_9 = \text{Min}\left[\frac{21}{2}, \frac{9}{2}, \frac{5}{2}\right] = \frac{5}{2}$$

$$B_1 = \text{Min}\left[\frac{5}{2} \times 1\right] = \frac{5}{2}$$

For B_2

$$H(B_2) = [9]$$

$$Q_9 = \text{Min}\left[\frac{21}{2}, \frac{21}{10}\right] = \frac{21}{10}$$

$$B_2 = \text{Min}\left[\frac{21}{10} \times 1\right] = \frac{21}{10}$$

$$\delta = \text{Min}\left[\gamma B_1, \gamma B_2\right]$$

$$= \text{Min}\left[\frac{5}{2} \times 1, \frac{21}{10} \times 1\right] = 21$$

B_2 gives a second best extreme point solution of N (II.1.3) by replacing f_7 by f_9.

$$X_3^{(2)} \quad X_2^{(3)} = \left[X_{31}^{(2)} = \left(\frac{39}{5}, \frac{18}{5}, \frac{38}{5}, \frac{3}{5}, \frac{36}{5}, \frac{12}{5}, 0, 0, \frac{21}{10}\right)\right]$$

$$J(X_{31}^{(2)}) = \left[d_1\ d_2, d_5, d_6\right]$$

$$J(x_{31}^{(2)}) = \left[d_1, d_2, d_5, d_6\right]$$

$$J(x_{31}^{(2)}) = 4 = p\ (= 4)$$

Also elements of $J(X_{31}^{(2)})$ are linearly independent.

$$X_3^{(2)} \cap S_1 \neq \emptyset$$

Hence $X_3^{(2)}$ is the solution of Problem N(II.1.1)

The solution for N (II.1.1) is

$$x_1 = \frac{39}{5}$$

$$x_2 = \frac{18}{5}$$

and optimal value of objective function of N(II.1.1) is

$$Z = \frac{132}{5}$$

SECTION 2

CUTTING PLACE PROCEDURE II

INTRODUCTION :

The cutting plane procedure I, given in Section I, has the disadvantages that

(i) It is required to test the linear independence of elements of $J(x)$ i.e. linear independence of a subject of columns of D which is computationally lengthy and difficult

(ii) The cuts introduced creat some additional extreme points which although are never going to be the solution of the problem (II.1.1) but are required to be studied.

The procedure presented in this section has the advantage that it eliminates the difficulty (i). Thus present technique is easier to handle and requires lesser computations and time. In this procedure one moves on the extreme points of $Dx = d, x \geq o$ startingfrom the first best and keep on ranking the extreme points until the feasibility in $Ax = b$ is satisfied. As soon as such an extreme point is obtained, it is optimal solution for the problem (II.1.1).

THEORITICAL DEVELOPMENT

Let $S = \{x : x$ is an extreme point of $Dx = d, x \geq o\}$ S is, obviously, a finite set. The problem (II.1.1) may be restated as

$$\begin{aligned} &\text{Max } Z = CX \\ &X \in s \\ &\text{subject to} \\ &\quad AX = b \\ &\quad X \geq 0 \end{aligned}$$

Then S is set of the extreme solutions of the problem

$$\left.\begin{aligned} &\text{Max } Z = CX \\ &\text{subject to} \\ &\quad DX = d \\ &\quad X \geq 0 \end{aligned}\right\} \qquad \text{(II.2.1)}$$

The problem (II.2.1) may be bounded or unbounded.

Case (i) :

The problem (II.2.1) is bounded. Introduce the following notations to develop the theory.

NOTATIONS :

$S_1 = \{ X : X \in S, AX = b \}$

$S = \{ X : X$ is an extreme point of $DX = d, X \geqslant 0 \}$

$X_i^{(1)}$ is the set of i^{th} best extreme point solutions of (II.2.1)

$U_i^{(1)}$ is the value of objective function corresponding to an element of $X_i^{(1)}$

$X_i^{(1)} = \{ X \in S : CX = U_i^{(1)} \}$

let $X_i^{(1)} = \{ X_{i1}^{(1)}, X_{i2}^{(1)} \ldots\ldots\ldots X_{ih_i}^{(1)} \}$

Apply simplex method to find the set of optimal solutions $X_1^{(1)}$ to the problem (II.2.1). If $X_1^{(1)} = \emptyset$, then problem (II.1.1) has no solution an process is terminated. In case $X_1^{(1)} \neq \emptyset$ then the elements of $X_1^{(1)}$ are tested one by one to see if they satisfy the feasibility in $AX = b$. If so then $X_1^{(1)} \cap S_1 \neq \emptyset$ and every element of $X_1^{(1)} \cap S_1$ is an optimal solution for (II.1.1) and procedure terminates. If $X_1^{(1)} \cap S_1 = \emptyset$ i.e. no element of $X_1{}^{(1)}$ satisfies the feasibility in $AX = b$, then determine $X_2^{(1)}$, the set of second best extreme point solutions of problem (II.2.1) (Procedure T discussed in Section I). If $X_2^{(1)} = \emptyset$, then problem (II.1.1) has no solution. If $X_2^{(1)} \neq \emptyset$, determine $X_2^{(1)} \cap S_1$, in case $X_2^{(1)} \cap S_1 \neq \emptyset$, every element of $X_2^{(1)} \cap S_1$ yields the optimal solution for (II.1.1). If $X_2^{(1)} \cap S_1 = \emptyset$ proceed to find $X_i^{(1)}$, the set of i^{th} best extreme point solution to (II.2.1), starting from $i = 3$, (Procedure developed in section I).

The process ends in either yielding $X_h^{(1)} = \emptyset$, for some h, implying problem (II.1.1) has no solution or $X_h^{(1)} \neq \emptyset$ and $X_h^{(1)} \cap S_1 \neq \emptyset$ which implies that every element of $X_h^{(1)} \cap S_1$ is optimal for problem (II.1.1); where $X_h^{(1)}$ is the set of h^{th} best extreme point solution of (II.2.1) and it will be the set of second best extreme point solution the problem :

$$\text{Max } Z = CX$$

subject to

$$DX = d$$

$$CX \quad U_{h-1}^{(1)}$$

$$X \geq 0$$

where $U_{h-1}^{(1)}$ is the value objective function at an element of $X_{h-1}^{(1)}$.

Hence the procedure, mainly works on three steps :

STEP - I : At the k^{th} iteration, the k^{th} best extreme point solutions of (II.2.1) is found.

STEP - 2 : The points obtained in STEP-I are texted for feasibility in $AX = b$.

STEP - 3 : (a) If the feasibility is satisfied the procedure is terminated.

(b) If the feasibility is not satisfied proceed to next iteration.

The procedure given above terminates in a finite number of steps as S is finite and extreme points found in STEP-I are never repeated.

Case (ii) : The problem (II.2.1) is unbounded. In this case the problem (II.2.1) is converted into a bounded problem by introducing an additional constraint $CX \leq M$, M is sufficiently large, positive finite number such that all the extreme points of (II.2.1) are included in the resulting region. The problem generated is

$$\begin{aligned} &\text{Max } Z = CX \\ &\text{subject to} \\ &\quad DX = d \\ &\quad CX \leq M \\ &\quad X \geq 0 \end{aligned}$$

Let $\underline{S} = X$ is an extreme point of $DX = d$

$CX \leq M$

$X \geq 0$

$X_1^{(o)}$ is the set of optimal solutions for problem (II.2.0)

$U_1^{(o)}$ is the value of objectivefunction at an element of $X_1^{(o)}$

$X_i^{(o)}$ is the set of i^{th} best extreme point solution of (II.2.0) i.e.

$X_i^{(o)} = X \in \underline{S} : CX = U_i^{(o)} \quad i = 1, \ldots. N$

$U_i^{(o)}$ is the value of objective function for an element of $X_i^{(o)}$

Since problem (II.2.0) is bounded, therefore $X_1^{(o)} \neq \emptyset$ and $CX \quad M \not\forall X \in S$ then $X_1^{(o)} \cap S_s = \emptyset$. The remaining extreme point solutions of (II.2.0) are ranked systematically till at some stage, say k^{th}, $X_k^{(o)} \cap S_1 \neq \emptyset$ in which case every element of $X_k^{(o)} \cap S_1$ is an optimal solution of (II.1.1).

The procedure developed in Section 2 is easier to handle in the sense that it is much easier to test if an element of S satisfies the feasibility in $AX = b$ rather than to test if an extreme point of

$FX = f$, $X \geq 0$ is also an extreme point of $DX = d$, $X \geq 0$.

Since the feasible region of problem (II.1.2) is more compact - restricted than the feasible region of the problem (II.2.1) of Section 2, therefore, $V_1^{(2)}$ is expected to be nearer the optimal value of the problem (II.1.1) than $U_1^{(1)}$ and therefore it is expected that procedure developed in Section 1 will move faster towards optimality of (II.1.1) but the computational advantage gained in the procedure of Section 2 helps to solve the problem in a lesser time.

In both the procedures developed in Section 1 and Section 2 if the optimal solution is not obtained at the first iteration itself, then second best extreme point solutions etc. are needed to be calculated. In general these are not unique and so all the alternate optimal solutions are to be calculated. The theorem that ensures that not all the simplex tables need to be considered and also in each table there is only one column with $Z_j - C_j \geq 0$ and $Q_j > 0$.

<u>THEOREM</u> : Each optimal extreme point solution of the problem (II.2.i), $i \geq 2$ is given by an optimal simplex table in which $Z_j - C_j = 0$ $i = 1, 2, \ldots, n$ and $Z_{n+1} - C_{n+1} = 1$.

PROOF : Consider the problem (II.2.1), introducing the slack variables it becomes

$$\text{Max } Z = CX$$

subject to

$$DX = d$$

$$CX + x_{n+1} = U_1^{(2)}$$

$$X \geq 0$$

let $$G = \begin{bmatrix} D & 0 \\ C & 1 \end{bmatrix}$$

$$h_1^{(2)} = \begin{bmatrix} d \\ U_1^{(2)} \end{bmatrix} \quad i \geq 2$$

The result is proved by using principle of induction consider the case when $i = 2$

The problem (II.2.2) is

$$\text{Max } Z = CX$$

subject to

$$GX = h_2^{(2)}$$

$$X \geq 0$$

let $X_{21}^{(2)}$ be an element of the set of second test extreme point solution of the problem (II.2.1). Let B be the basis corresponding to this element, then simplex table for this solution will have at least one $Z_j - C_j < 0$

$j = 1, 2, \ldots . n.$ Consider a basis B_1 of the problem (II.2.2) given as

$$B_1 = \begin{bmatrix} B & 0 \\ C_B & 1 \end{bmatrix}$$

Then $$B_1^{-1} = \begin{bmatrix} B^{-1} & 0 \\ -C_B B^{-1} & 1 \end{bmatrix}$$

and $$C_{B_1} = [\, C_B, \ 0 \,]$$

The vector X_{B_1} of the basic variables for the problem (II.2.2) for the basis B_1 is given by

$$\begin{aligned} X_{B_1} &= B^{-1} h_2^{(2)} \\ &= \begin{bmatrix} B^{-1} & 0 \\ -C_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} d \\ U_2^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} B^{-1} d \\ -U_2^{(2)} + U_2^{(2)} \end{bmatrix} \\ &= \begin{bmatrix} X_{21}^{(2)} \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X_j &= B g_j^{-1} \\ &= \begin{bmatrix} B^{-1} d_j \\ -C_B B^{-1} d_j + C_j \end{bmatrix} \qquad j = 1, 2 \ldots n \\ & \begin{bmatrix} B^{-1} & 0 \\ -C_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad j = n + 1 \end{aligned}$$

$$= \begin{cases} \begin{bmatrix} B^{-1} d_j \\ -Z_j + C_j \end{bmatrix} & j = 1, 2, \ldots. n \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & j = n + 1 \end{cases}$$

$$\hat{Z}_j = C_{B_1} \hat{Y}_j$$

$$\hat{Z}_j - C_j = C_{B_1} \hat{Y}_j - C_j$$

$$= \begin{cases} \left[[C_B, 0] \begin{bmatrix} B^{-1} d_j \\ - Z_j + C_j \end{bmatrix} - C_j \right] & j = 1,2, \ldots n \\ [C_B \;\; 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} & j = n + 1 \end{cases}$$

$$= \begin{cases} [C_B B^{-1} d_j - C_j] & j = 1,2, \ldots n \\ 0 & j = n + 1 \end{cases}$$

$$Z_j - C_j = \begin{cases} Z_j - C_j & j = 1,2, \ldots. n \\ 0 & j = n + 1 \end{cases}$$

Thus for the problem (II.2.2) all values of $Z_j - C_j$ are the same as for the problem (II.2.1) $j=1,2,.. n$

and since slack variable is in the basis its corresponding $Z_j - C_j = 0$. Since in the simplex table corresponding to $Z_{21}^{(2)}$ there is at least one column say h^{th} for which $Z_h - C_h < 0$. Therefore, $\hat{Z}_h - C_h = Z_h - C_h < 0$ and hence h^{th} column enters the basis also since x_{n+1} is at zero level in the basis therefore $_{n+1}$ is removed in the process of simplex iteration. In the resulting simplex table $Z_j - C_j$ is given by

$$Z_j - C_j = (\hat{Z}_j - C_j) - (\hat{Z}_h - C_h) \frac{\hat{y}_{(n+1)j}}{\hat{y}_{(n+1)h}} \quad j=1,2\ldots n$$

$$= (Z_j - C_j) - (Z_h - C_h) \frac{(Z_j + C_j)}{(-Z_h + C_h)} \quad j=1,2,\ \ldots\ n$$

$$\tilde{Z}_j - C_j = 0 \quad j = 1,2,\ \ldots\ n$$

$$\tilde{Z}_{+1} - C_{n+1} = 0 - (Z_h - C_h) \frac{1}{(-Z_h + C_h)}$$

$$= 1$$

Hence $\tilde{Z}_j - C_j = 0$ if $j = 1,2,\ \ldots.\ n$

$= 1$ if $j = n + 1$

Optimality criteria is satisfied.

Hence the result

The other optimal simplex tables can be generated starting with this table. It is observed that any $_j$, $j = 1,2\ \ldots.\ n$, not in the basis, if entered into the basis preserves this character of $Z_j - C_j$ also $_{n+1}$

can be brought into the basis only at zero level if the optimality is to be maintained and hence no new optimal extreme point solution is obtained.

Now it is shown that it is not necessary to bring $_{n+1}$ into the basis to obtain δ where

$$\delta = \text{Min}\left\{ \gamma_B \text{ where B is a basis for the element } x_{21}^{(2)} \right\}$$

$$\gamma_B = \text{Min}\left\{ \theta_j \ (z_j - c_j); \quad Q_j > 0 \right\}$$

$$H\ (B) = \left\{ j : z_j - c_j > 0 \right\}$$

Suppose $\delta = \theta_h \ (z_h - c_h) = \dfrac{X_{Br}}{y_{rh}} (z_h - c_h)$

for some basis B containing q_{n+1}

This gives a representation of an extreme point of $DX = d,\ x \geq o$ wherein $x_{n+1} = X_{B\ (m+1)} = 0$ then $Y_{(n+1)h} = Z_k + C_k \neq 0$ and thus $_k$ can be entered departing $_{n+1}$. As a result of this iteration in the new simplex table the enteries are

$$\bar{X}_{Bi} = X_{Bi} \quad i = 1,2, \ldots\ldots m+1$$

$$\bar{S} = \begin{cases} \dfrac{X_{Bi}}{Y_{ih}} (Z_h - C_h) & i = 1,2, \ldots\ldots m \\ 0 & i = m + 1 \end{cases}$$

$$\tilde{Z}_j - c_j = \begin{cases} 0 & j = 1,2 \ldots n \\ 1 & j = n+1 \end{cases}$$

Hence $\tilde{C}$ is given by an optimal simplex table in which the basis does not contain g_{n+1}. Hence it is unnecessary to consider g_{n+1} for entry into basis for finding optimal.

This completes the proof for i=2. Suppose now the result is true for i=2,3, ... h - 1 Then any second best extreme point solution to the problem (II.2.h-1) must have g_{n+1} in the basis and the basis is of the form.

$$\bar{B}_1 = \begin{bmatrix} B & 0 \\ C_B & 1 \end{bmatrix}$$

and

$$\bar{B}_1^{-1} = \begin{bmatrix} B^{-1} & 0 \\ -C_B B^{-1} & 1 \end{bmatrix}$$

$$X_{\bar{B}_1} = \begin{bmatrix} B^{-1} & 0 \\ -C_B B^{-1} & 1 \end{bmatrix} \begin{bmatrix} d \\ U_h^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} B^{-1}d \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} X_{B_h} \\ 0 \end{bmatrix} \qquad X_{B_h}$$

Repeating the arguments given for i=2, we get the result.

This theorem helps in reducing the number of optimal simplex tables of the problem (II.2.i), $i \geq 2$ since g_{n+1} is never to be brought into the basis. Also the calculations of second best extreme point solution of this problem are greatly reduced since $H(B) = \{n+1\}$ i.e. singleton and also $Z_{n+1} - C_{n+1} = 1$.

EXAMPLE

$$\left.\begin{array}{l}
\text{Max } Z = 7x_1 + 9x_2 \\
\text{subject to} \\
\quad -x_1 + 2x_2 \leq 1 \qquad (a.1) \\
\quad 2x_1 + 9x_2 \leq 1 \qquad (a.2) \\
\quad x_1, x_2 \geq 0 \\
\text{and } (x_1, x_2) \text{ is extreme point of} \\
\quad -2x_1 + x_2 \leq 1 \\
\quad x_2 \leq 4 \\
\quad x_1 + x_2 \leq 7 \\
\quad x_1 - 4x_2 \leq 2 \\
\quad x_1, x_2 \geq 0
\end{array}\right| \qquad N(II.2.1)$$

<u>SOLUTION</u> :

In order to solve N(II.2.1) consider the following problem :

$$\left.\begin{array}{l} \text{Max } Z = 7x_1 + 9x_2 \\ \text{subject to} \\ -2x_1 + x_2 + x_3 = 1 \\ \quad x_2 + x_4 = 4 \\ x_1 + x_2 + x_5 = 7 \\ x_1 - 4x_2 + x_6 = 2 \\ x_1, x_2, \ldots, x_6 \; 0 \end{array}\right|$$

$DX = d$
$X \geq 0$

The optimal solution of N(II.2.2) is

$$X_1^{(2)} = \left[X_{11}^{(2)} = (3,4,3,0,0,150 \right]$$

$$U_1^{(2)} = 51$$

$$X_1^{(2)} \cap S_1 = \emptyset$$

To find out the set $X_2^{(2)}$ of second best extreme point solutions of N(II.2.2), let B be the basis for $X_1^{(2)}$.

Now $H(B) = \{4, 5\}$

$\theta_4 = \text{Min } [4, 3] = 3$

$\theta_5 = \text{Min } [3, \frac{3}{2}] = \frac{3}{2}$

$$B = \text{Min} \quad 3 \times 2,\ 3/2 \times 7 \quad = \quad 6$$

$$\delta \quad = \quad 6$$

Thus replacing d_6 by d_5 in B a second best extreme point solution of N(II.2.2) is obtained

$$X_2^{(2)} = \left[X_{21}^{(2)} = (6,1,12,3,0,0,0) \right]$$

$$U_2^{(2)} = 5.1$$

Simplex Table For $X_{21}^{(2)}$

C_B	variable of the basis	C_j / X_B	7 / d_1	9 / d_2	0 / d_3	0 / d_4	0 / d_5	0 / d_6
9	d_2	1	0	1	0	0	1/5	-1/5
7	d_1	6	1	0	0	0	4/5	1/5
0	d_3	12	0	0	1	0	1/5	3/5
0	d_4	3	0	0	0	1	-1/5	1/5
	Z=51	$Z_j - C_j$	0	0	0	0	37/5	-2/5

Now $\qquad X_2^{(2)} \cap S_1 = \emptyset$

Introduce the cut $7x_1 + 9x_2 \quad 51$ in (II.2.2) the problem so obtained is

$$\begin{aligned}
&\text{Max} \quad Z = 7x_1 + 9x_2 \\
&\text{subject to} \\
&-2x_1 + x_2 + x_3 = 1 \\
&x_2 + x_4 = 4 \\
&x_1 + x_2 + x_5 = 7 \\
&x_1 - 4x_2 + x_6 = 2 \\
&7x_1 + 9x_2 + x_7 = 51 \\
&x_1, x_2, \ldots x_7 = 51 \\
&x_1, x_2, \ldots x_7 \geq 0
\end{aligned} \qquad N\,(II.2.3)$$

The set of optimal solutions for N(II.2.3) is

$$X_1^{(3)} = \left\{ X_{11}^{(3)} = \left(\frac{15}{7}, 4, \frac{9}{7}, 0, \frac{6}{7}, \frac{111}{7}, 0 \right) \right.$$

$$\left. X_{12}^{(3)} = (6, 1, 12, 3, 0, 0, 0) \right\}$$

let B_1 be basis for $X_{11}^{(3)}$ and B_2 be basis for $X_{12}^{(3)}$

For B_1 :

$$H(B_1) = \{ 7 \}$$

$$\theta_7 = \text{Min} \left\{ 15, \frac{9}{2} \right\} = \frac{9}{2}$$

$$B_1 = \frac{9}{2}$$

For B_2 :

$$H(B_2) = 7$$

$$\theta_7 = \text{Min}\left[2,0,8\right] = 0$$

Hence $\delta = \frac{9}{2}$

Therefore, second best extreme point solution of N(II.2.3) is obtained by replacing d_3 by d_7 in B_1.

$$X_3^{(2)} \quad X_2^{(3)} = X_{31}^{(2)} = \left(\frac{3}{2}, 4, 0, 0, \frac{3}{2}, \frac{33}{2}, \frac{9}{2}\right)$$

$$U_3^{(2)} = \frac{93}{2}$$

$$X_3^{(2)} \cap S_1 = \emptyset$$

Now

Introduce the cut $7x_1 + 9x_2 \leq \frac{93}{2}$ in N(II.2.2)

The following problem is obtained

$$\begin{aligned}
&\text{Max } Z = 7x_1 + 9x_2 \\
&\text{subject to} \\
&-2x_1 + x_2 + x_3 = 1 \\
&x_2 + x_4 = 4 \\
&x_1 + x_2 + x_5 = 7 \\
&x_1 - 4x_2 + x_6 = 2 \\
&7x_1 + 9x_2 + x_8 = \frac{93}{2} \\
&x_1, x_2, \ldots, x_6, x_8 \geq 0
\end{aligned} \qquad \text{N(II.2.4)}$$

The optimal solution for N(II.2.4) is

$$\left[X_1^{(4)} = X_{11}^{(4)} = \left(\frac{3}{2}, 4, 0, 0, \frac{3}{2}, \frac{33}{2}, 0 \right) \right.$$

$$\left. X_{12}^{(4)} = \left(\frac{204}{37}, \frac{65}{74}, \frac{825}{74}, \frac{231}{74}, \frac{45}{74}, 0, 0 \right) \right]$$

let B_3 be basis for $X_{11}^{(4)}$ and B_4 be basis for $X_{12}^{(4)}$

For B_3 :

$$H(B_3) = [8]$$

$$\theta_8 = \text{Min}\left[\frac{75}{2} \quad \frac{33}{2} \times \frac{25}{7}, 50 \right] = \frac{75}{2}$$

$$\ell B_3 = \frac{75}{2}$$

For B_4 :

$$H(B_4) = [8]$$

$$\theta = \text{Min}\left[\frac{65}{2} \quad \frac{825}{14}, 51 \right] = \frac{65}{2}$$

$$\ell B_4 = \frac{65}{2}$$

$$\delta = \text{Min}\left[\frac{75}{2}, \frac{65}{2} \right] = \frac{65}{2}$$

∴ Replacing d_2 by d_3 in B_4 a second best extreme point solution of N(II.2.4) is obtained

$$X_4^{(2)} = X_2^{(4)} = \left[X_{41}^{(2)} = \left(2, 0, \frac{185}{37}, \frac{49}{37}, \frac{185}{37}, 0, \frac{65}{2} \right) \right]$$

$$U_2^{(4)} = 14$$

Now $X_2^{(4)}$ satisfies both (a.1) and (a.2)

Therefore

$$x_1 = 2$$

$$x_2 = 0$$

is optimal solution for N(II.2.1) and $Z = 14$ is optimal value of objective function.

SECTION - 3
ENUMERATION TECHNIQUES FOR SOLVING EPLPP

In Section I of this chapter a cutting place procedure is presented to solve the problem (II.1.1). The procedure moves over the extreme points of $FX = f, X \geq 0$ in decreasing order of the value of the objective function till an extreme point of $DX = d, X \geq 0$ is achieved to obtain second best , 3rd best etc. extreme point solutions cuts were introduced which gave rise to alternate solutions which were not extreme points of $FX = f, X \geq 0$. The enumeration technique presented in this section removes this difficulty.

In order to solve problem (II.1.1), consider problem (II.1.2) which is

$$\text{Max } Z = CX$$

subject to

$$FX = f$$

$$X \geq 0$$

let $X_i^{(2)} = \{ X_{i1}^{(2)}, X_{i2}^{(2)} \ldots X_{ik_i}^{(2)} \}$ be the set of i^{th} best extreme point solutions of (II.1.2) and $V_i^{(2)}$ be the value of objective function at an element of $X_i^{(2)}$.

$$B_i^{(2)} = \{ B_{i1}^{(2)}, B_{i2}^{(2)} \ldots B_{ie_i}^{(2)} \} \; ; \; li \geq k_i$$

be the set of bases corresponding to elements of $X_i^{(2)}$

$$E_i^{(2)} = \{ E_{i1}^{(2)}, E_{i2}^{(2)} \ldots E_{im_i}^{(2)} \}$$ be the set of

bases adjacent to the elements of $B_i^{(2)}$ yielding the value of the objective function less than $V_i^{(2)}$. Apply simplex method to find $X_1^{(2)}$ and $V1^{(2)}$. If $X_1^{(2)} = \emptyset$, then (II.1.2) has no solution and hence (II.1.1) has no solution. If $X_1^{(2)} \neq \emptyset$, then determine $X_1^{(2)} \cap S_1$, in case $X_1^{(2)} \cap S_1 \neq \emptyset$ then every element of $X_1^{(2)} \cap S_1$ is optimal solution for (II.1.1). Otherwise, find the sot $X_2^{(2)}$ as follows: Determine $B_1^{(2)}$ and $E_1^{(2)}$. The subset $B_2^{(2)}$ of $E_1^{(2)}$ which yield the greatest value of the objective function, say $V_2^{(2)}$, generates the set $X_2^{(2)}$. If $X_2^{(2)} \neq \emptyset$ and $X_2^{(2)} \cap S_1 \neq \emptyset$ then every element of $X_2^2 \cap S_1$ is a solution of (II.1.1). Otherwise proceed to find $X_2^{(3)}$ as follows:

$$\text{let } H_1^{(2)} = E_1^{(2)}$$

$$H_2^{(2)} = (E_1^{(2)} \cup E_2^{(2)}) - B_2^{(2)}$$

The subset $B_3^{(2)}$ of $H_2^{(2)}$ that yields the greatest value of the objective function, say $V_3^{(2)}$, generates the set $X_3^{(2)}$. If $X_3^{(2)} = \emptyset$ then (II.1.1) has no solution. If $X_3^{(2)} \neq \emptyset$ and $X_3^{(2)} \cap S_1 \neq \emptyset$ then every element of $X_3^{(2)} \cap S_1$ is optimal for (II.1.1). Otherwise, proceed to find the next best extreme point solution.

At the i^{th} iteration find the set $E_{i-1}^{(2)}$ of bases adjacent to the elements of $B_{i-1}^{(2)}$ yielding the value of objective function less than $V_{i-1}^{(2)}$. Determine

$$H_{i-1}^{(2)} = \left[\bigcup_{j=1}^{i-1} E_j^{(1)} \right] - \left[\bigcup_{j=2}^{i-1} B_j^{(2)} \right]$$

A subset $B_i^{(2)}$ of $H_{i-1}^{(2)}$ yielding greatest value, say $V_i^{(2)}$, of the objective function generate the set $X_i^{(2)}$

This process of finding next best extreme point solution is continued till either $X_h^{(2)} = \emptyset$, for some h, indicating that (II.1.1) has no solution (This will be so when at that particular iteration $H_{h-1}^{(2)} = \emptyset$) or $X_h^{(2)} \neq \emptyset$ and $X_h^{(2)} \cap S_1 \neq \emptyset$ in which case every element of $X_h^{(2)} \cap S_1$ is optimal solution for (II.1.1).

The procedure converges in a finite number of steps as

(i) $S_1 \subseteq S_2$ and S_2 is finite

(ii) No extreme point solution is repeated since

$$V_i^{(2)} > V_{i+1}^{(2)} \quad \forall\ i$$

Consider the case when (II.1.2) has a solution but (II.1.1) has no solution. In this case $X_i^{(2)} \cap S_1 = \emptyset \ \forall i$. Thus after a finite number of steps say N, it will be impossible to find out (N + 1)th best extreme point solution i.e. $H_N^{(2)} = \emptyset$.

PROBLEM :

Here problem N(II.1.1) is solved by enumeration technique presented in Section 3.

SOLUTION :

The optimal solution for N(II.1.2) is

$$X_1^{(2)} = \left[X_{11}^{(2)} = 6,6,2,0,3,0,6,3)\right]$$

$$U_1^{(2)} = 30$$

Now $X_1^{(2)} \cap S_1 = \emptyset$

$$B_1^{(2)} = \left[B_{11}^{(2)} = (f_1, f_3, f_2, f_5, f_7, f_8)\right]$$

$$E_1^{(2)} = \left[E_{11}^{(2)} = (f_1, f_4, f_2, f_5, f_7, f_8) \right.$$

$$\left. E_{12}^{(2)} = (f_1, f_3, f_2, f_5, f_7, f_8) \right]$$

Value of objective function for

$$E_{11}^{(2)} = 30 - \frac{2}{1} \times 2 = 26$$

$$E_{12}^{(2)} = 30 - \frac{3}{2} \times 1 = 30 - 1.5 = 28.5$$

Therefore, $E_{12}^{(2)}$ generates an element of $X_2^{(2)}$

$$X_2^{(2)} = \left[X_{21}^{(2)} = (\frac{15}{2}, \frac{9}{2}, \frac{13}{2}, 0, 6, \frac{3}{2}, \frac{3}{2}, 0) \right]$$

Now $X_2^{(2)} \cap S_1 = \emptyset$

$$B_2^{(2)} = \left[B_{21}^{(2)} = (f_1, f_3, f_2, f_5, f_7, f_6) \right]$$

$$E_2^{(2)} = \left[E_{21}^{(2)} = (f_1, f_3, f_2, f_5, f_4, f_6) \right]$$

$$H_2^{(2)} = \left[H_{21}^{(2)} = (f_1, f_3, f_2, f_5, f_4, f_6), \right.$$

$$\left. H_{22}^{(2)} = H_{21}^{(2)} = (f_1, f_4, f_2, f_5, f_7, f_8) \right]$$

Value of objective function for

$$H_{21}^{(2)} = \frac{57}{2} - \frac{3}{2} \times \frac{2}{5} \times \frac{7}{2} = \frac{264}{10}$$

$$H_{22}^{(2)} = 30 - 2 \times 2 = 26$$

Therefore $H_{21}^{(2)}$ generates an element of $X_3^{(2)}$

$$X_3^{(2)} = \left[X_{31}^{(2)} = \left(\frac{39}{5}, \frac{18}{5}, \frac{43}{5}, \frac{3}{5}, \frac{36}{5}, \frac{12}{5}, 0, 0,\right)\right]$$

$$U_3^{(2)} = \frac{132}{5}$$

$$X_3^{(2)} \cap S_1 \neq \emptyset$$

Therefore, $x_1 = \frac{39}{5}$, $x_2 = \frac{18}{5}$ is required solution of N(II.1.1) and

$Z = \frac{132}{5}$ is optimal value of objective function.

SECTION - 4
ANOTHER ENUMERATION TECHNIQUE

In Section 2 a cutting place algorithm was presented which removes the difficulty of testing the linear independence of a subset of columns of D. But in this case the problem of calculating alternate solutions remained. The Enumerative Procedure presented here eliminates this difficulty too. Thus this approach will be best one for solving an EPLPP.

$$\text{Max } Z = CX$$

subject to

$$DX = d$$

$$X \geq 0$$

let $U_i^{(1)}$ be the value of objective function of (II.2.1) at an element of $X_i^{(1)}$, the set of i^{th} best extreme point solution of (II.2.1)

$$X_i^{(1)} = \{ X_{i1}^{(1)}, X_{i2}^{(1)} \ldots\ldots X_{ih_i}^{(1)} \}$$

$$= \{ X \in S \; ; \; CX \quad U_i^{(1)} \}$$

$$B_i^{(1)} = \{ B_{i1}^{(1)}, \; B_{i2}^{(1)}, \ldots\ldots \; B_{il_i}^{(1)} \}$$

$$E_i^{(1)} = \{ E_{i1}^{(1)}, E_{i2}^{(1)}, \ldots\ldots \; E_{im_i}^{(1)} \}$$

$B_i^{(1)}$ is the set of bases corresponding to elements of $X_i^{(1)}$ and $E_i^{(1)}$ is the set of bases adjacent to elements of $B_i^{(1)}$ yielding value of objective function less than $U_i^{(1)}$.

$$H_i^{(1)} = \left[\bigcup_{j=1}^{i} E_j^{(1)} \right] - \left[\bigcup_{i=2}^{i} B_j^{(1)} \right]$$

is the set of bases adjacent to first best, second best ..... i^{th} best extreme point solution minus the bases corresponding to first best, second best, ... i^{th} best extreme point therefore, the set of bases of the elements of $X_{i+1}^{(1)}$ must be a subset of $H_i^{(1)}$ i.e. $B_{i+1}^{(1)} \subseteq H_i^{(1)}$.

Problem (II.2.1) is assumed to be bounded because in case it is unbounded it can be converted into a bounded problem by introducing a constraint of the form $CX \leq M$, where M is sufficiently large positive finite number, as discussed in Section I of this chapter.

Determine the set $X_1^{(1)}$. If $X_1^{(1)} = \emptyset$ problem (II.1.1) has no solution and procedure is terminated. In case $X_1^{(1)} \neq \emptyset$, then test if any element of $X_1^{(1)}$ satisfies the feasibility in $AX = b$ i.e. determine $X_1^{(1)} \cap S_1$. In case $X_1^{(1)} \cap S_1 \neq \emptyset$ then every element of $X_1^{(1)} \cap S_1$ is an optimal solution for problem (II.1.1). Otherwise the extreme point solutions of (II.2.1) are enumerated (Technique given in Section 3) in a systematic order till at some stage, say h^{th}, we get either $X_h^{(1)} = \emptyset$ which is indicated by $H_{h-1}^{(1)} = \emptyset$ implying that (II.1.1) has no solution or $X_h^{(1)} \neq \emptyset$, $X_h^{(1)} \cap S_1 \neq \emptyset$ in which case every element of $X_h^{(1)} \cap S_1$ is a solution of problem (II.1.1).

<u>EXAMPLE</u> :

Here problem N(II.2.1) is solved by enumeration technique developed in Section 4.

The optimal solution for N(II.2.2) is

$$X_1^{(2)} = \{ X_{11}^{(2)} = (3, 4, 3, 0, 0, 15) \}$$

$$U_1^{(2)} = 57$$

$$B_1^{(2)} = \left[B_{11}^{(2)} = (d_2, d_1, d_3, d_6) \right]$$

$$E_1^{(2)} = \left[E_{11}^{(2)} = (d_2, d_1, d_3, d_4), \right.$$

$$\left. E_{12}^{(2)} = (d_2, d_1, d_5, d_6) \right]$$

Value of the objective function for

$$E_{11}^{(2)} = 57 - \frac{15}{5} \times 2 = 51$$

$$E_{12}^{(2)} = 57 - \frac{13}{2} \times 7 = \frac{93}{2} = 46.50$$

Therefore $E_{11}^{(2)}$ generates $X_2^{(2)}$ the set of second best extreme point solutions of $N(II.2.2)$

$$X_2^{(2)} = \left[X_{21}^{(2)} = (6,1,12,3,0,0,0) \right]$$

$$U_2^{(2)} = 51$$

$$X_2^{(2)} \cap S_1 = \emptyset$$

and

$$B_2^{(2)} = \left[(d2, d1, d3, d4) = B_{21}^{(2)} \right]$$

$$E_2^{(2)} \quad \left[(d_5, d_1, d_3, d_4) = E_{21}^{(2)} \right]$$

$$H_2^{(2)} = \left[H_{21}^{(2)} = (d_2, d_1, d_5, d_6), \right.$$

$$\left. H_{22}^{(2)} = (d_5, d_1, d_3, d_4) \right]$$

Value of objective function for

$$H_{21}^{(2)} = \frac{93}{2}$$

$$H_{22}^{(2)} = 14$$

Therefore, $H_{21}^{(2)}$ generates $X_3^{(2)}$, the set of third best extreme point solutions of H(II.2.2).

$$X_3^{(2)} = \left[X_{31}^{(2)} = \left(\frac{3}{2}, 4, 0, 0, \frac{3}{2}, \frac{33}{2} \right) \right]$$

$$U_3^{(2)} = \frac{93}{2}$$

Now

$$B_3^{(2)} = \left[B_{31}^{(2)} = (d_2, d_1, d_5, d_6) \right]$$

$$E_3^{(2)} = \left[E_{31}^{(2)} = (d_2, d_4, d_5, d_6) \right]$$

$$H_3^{(2)} = \left[E_1^{(2)} \quad UE_2^{(2)} \ UE_3^{(2)} \quad - \quad B_2^{(2)} \ UB_3^{(2)} \right]$$

$$H_3^{(2)} = \left[H_{31}^{(2)} = (d_2 \ d_4, \ d_5, d_6), \right.$$

$$\left. H_{32}^{(2)} = (d_5, d_1, d_3, d_4) \right]$$

Value of the objective function for

$$H_{31}^{(2)} = 9$$

$$H_{32}^{(2)} = 14$$

Therefore, $H_{32}^{(2)}$ generates $X_4^{(2)}$ and

$$X_4^{(2)} = \left[X_{41}^{(2)} = (2, 0, 5, 5) \right]$$

$X_4^{(2)}$ satisfies (a.1) and (a.2), therefore

$$x_1 = 2$$

$$x_2 = 0$$

is optimal solution for N(II.2.1) and

Z = 14 is optimal value of objective function.

CHAPTER - III

IMPROVED TECHNIQUE FOR SOLVING E P L P P

This chapter develops two cuts termed as 'Deep Cut' and 'Strong Cut' which make the procedure for solving EPLPP computationally more efficient over the procedures presented in Chapter II in the sense that the study of a number of extreme points of $DX=d, X \geq 0$ is avoided. Section I and Section 2, of this chapter present the cutting plane techniques while Section 3 deals with enumeration technique.

SECTION - I

DEEP CUT CUTTING PLANE PROCEDURE FOR SOLVING EPLPP

INTRODUCTION:

This procedure is an improvement over the procedures for solving an EPLPP, presented in Section 2, Chapter II.

THEORETICAL DEVELOPMENT :

Consider the problem (II.1.2) viz.

$$\text{Max } Z = CX$$

subject to

$$FX = f$$

$$X \geq 0$$

$$F = \begin{bmatrix} A \\ D \end{bmatrix}, \quad f = \begin{bmatrix} b \\ d \end{bmatrix}$$

This problem is assumed to be bounded because if it is not bounded it can always be converted into a bounded one by introducing an additional constraint $CX \leq M$ as discussed in Chapter II.

Apply simplex method to (II.1.2) to find $Y_1^{(2)}$, the set of optimal solutions of (II.1.2). If $Y_1^{(2)} = \emptyset$, then problem (II.1.1) has no solution and procedure is terminated. If $Y_1^{(2)} \neq \emptyset$ then determine $Y_1^{(2)} \cap S_1$ (Procedure given in Section I, Chapter II). If $Y_1^{(2)} \cap S_1 \neq \emptyset$ then every element of $Y_1^{(2)} \cap S_1$ is optimal solution for (II.1.1). If $Y_1^{(2)} \cap S_1 = \emptyset$ then find the set $Y_2^{(2)}$ of second best extreme point solution of (II.1.2) (Procedure given in Section I, Chapter II). If $Y_2^{(2)} = \emptyset$ then problem (II.1.1) has no solution and procedure is terminated. If $Y_2^{(2)} \neq \emptyset$, determine $Y_2^{(2)} \cap S_1$. If $Y_2^{(2)} \cap S_1 \neq \emptyset$ then every element of $Y_2^{(2)} \cap S_1$ is optimal for problem (II.1.1). In case $Y_2^{(2)} \cap S_1 = \emptyset$, let $V_2^{(2)}$ be the value of objective function for an element of $Y_2^{(2)}$.

Now pick up the problem (II.2.1) viz.

$$\begin{aligned} \text{Max } Z &= CX \\ \text{subject to} & \\ DX &= d \\ X &\geq 0 \end{aligned}$$

In case (II.2.1) is unbounded addition of a suitable constraints $CX \leq M$ as mentioned earlier makes it bounded. Find $X_1^{(1)}$ the set of optimal solutions of (II.2.1). Let $U_1^{(1)}$ be the value of objective function at an element of $X_1^{(1)}$. If $X_1^{(1)} = \emptyset$ then problem (II.1.1) has no solution and procedure terminates. Let $X_1^{(1)} \neq \emptyset$, since the feasible region of (II.2.1) contains the feasible region of (II.1.2). Therefore, $V_1^{(2)} \leq U_1^{(1)}$ (equality holds in case $Y_1^{(2)} \cap X_1^{(1)} \neq \emptyset$, where $V_1^{(2)}$ is the value of objective function of (II.1.2) for an element of $Y_1^{(2)}$. As $Y_1^{(2)} \cap S_1 = \emptyset$, $V_1^{(2)}$ is not optimal for (II.1.1) and $V_1^{(2)} \leq U_1^{(1)}$ therefore, $U_1^{(1)}$ is not optimal for (II.1.1). Hence $X_1^{(1)} \cap S_1 = \emptyset$. Now find the set of extreme points adjacent to the elements of the set $S_1^{(1)}$ for which the value of the objective function is greater than or equal to $V_2^{(2)}$. Out of these values, of the objective function, pick up the value V which is just greater than or equal to $V_2^{(2)}$, $V < U_1^{(1)}$. As $V \geq V_2^{(2)}$ and $Y_2^{(2)} \cap S_1 = \emptyset$, V will not be optimal for (II.1.1).

Introduce the cut $CX \leq V$, termed as 'deep cut' in problem (II.2.1) and solve the problem

$$\left.\begin{array}{l} \text{Max } Z = CX \\ \text{subject to} \\ \quad DX = d \\ \quad CX \leq V \\ \quad X \geq 0 \end{array}\right\} \qquad (\text{III.1.3})$$

Apply simplex method to find $X_1^{(3)}$, the set of optimal extreme point solution of (III.1.3). As $X_1^{(3)}$ is not optimal for (III.1.1) find $X_2^{(3)}$, the set of second best extreme point solutions of (III.1.3) (Procedure given in Section 2, Chapter III). If $X_2^3 = \emptyset$ then problem (III.1.1) has no solution. If $X_2^{(3)} \neq \emptyset$ then find $X_2^{(3)} \cap S_1$. If $X_2^{(3)} \cap S_1 \neq \emptyset$, then every element of $X_2^{(3)} \cap S_1$ is optimal for (II.1.1). If $X_2^{(3)} \cap S_1 = \emptyset$, the remaining extreme points solutions of (III.1.3) are ranked by cutting plane technique (Procedure given in Section 2, Chapter II) till either an optimal solution of (II.1.1) reached or an indication of no solution for (II.1.1) is indicated.

<u>Example</u> :

$$
\left.
\begin{array}{l}
\text{Max } \; Z = 8x_1 + 23x_2 \\
\text{subject to} \\
\quad -x_1 + 2x_2 \leq 10 \qquad (a.1) \\
\quad x_1 + 2x_2 \leq 14 \qquad (a.2) \\
\quad x_1 , x_2 \geq 0 \\
\text{and } (x_1, x_2) \text{ is extreme point of} \\
\quad -x_1 + x_2 \leq 7 \\
\quad -2x_1 + x_2 \leq 4 \\
\quad -5x_1 + 9x_2 \leq 90 \\
\quad 25x_1 + 27x_2 \leq 675
\end{array}
\right| \quad N(III.1.1)
$$

$$\begin{aligned} 21x_1 + 11x_2 &\leq 462 \\ x_2 &\leq 15 \\ 5x_1 - 6x_2 &\leq 60 \\ x_1 - 4x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

SOLUTION :

In order to solve N(III.1.1) start with the problem

$$\text{Max} \quad Z = 8x_1 + 23x_2$$

subject to

$$FX = f, \quad X \geq 0 \qquad \begin{aligned} -x_1 + x_2 + x_3 &= 10 \\ x_1 + 2x_2 + x_4 &= 14 \\ -x_1 + x_2 + x_5 &= 7 \\ -2x_1 + x_2 + x_6 &= 4 \\ -5x_1 + 9x_2 + x_7 &= 90 \\ 25x_1 + 27x_2 + x_8 &= 675 \\ 21x_1 + 11x_2 + x_9 &= 462 \\ x_2 + x_{10} &= 15 \\ 5x_1 - 6x_2 + x_{11} &= 60 \\ x_1 - 4x_2 + x_{12} &= 8 \\ x_1, x_2, \ldots, x_{12} &\geq 0 \end{aligned} \qquad N\ (III.1.2)$$

The optimal extreme point solution of N(III.1.2) is

$$Y_1^{(2)} = \{ Y_{11}^{(2)} = (2,6,0,0,3,2,4,6,463,360,9,87,30) \}$$

$$V_1^{(2)} = 154$$

$$J = \{ d_1, d_2, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12} \}$$

$$p = 8$$

$$| J (Y_{11}^{(2)}) | = 10 > 8$$

$$\therefore Y_1^{(2)} \cap S_1 = \emptyset$$

Find $Y_2^{(2)}$, the set of second best extreme point solutions of N(III.1.2). Let B be basis for $Y_{11}^{(2)}$.

Now

$$H (B) = \{3, 4\}$$

$$\theta_3 = 20$$

$$\theta = \frac{8}{3}$$

$$B = \text{Min} \left\{ 20 \times \frac{7}{4} , \frac{8}{3} \times \frac{39}{4} \right\}$$

$$= \text{Min} \{ 35, 26 \} = 26$$

$$\therefore \delta = 26$$

Replacing f_6 by f_5 in B, $Y_2^{(2)}$ is obtained

$$Y_2^{(2)} = \left\{ Y_{21}^{(2)} = \left(\frac{2}{3}, \frac{16}{3}, 0, \frac{8}{3}, \frac{8}{3}, 0, \frac{136}{3}, \frac{1543}{3}, \frac{1186}{3}, \frac{29}{3}, \frac{269}{3}, \frac{86}{3} \right) \right\}$$

$$V_2^{(2)} = 128$$

$$J(X_{21}^{(2)}) = 9 > 8$$

$$X_2^{(2)} \cap S_1 = \emptyset$$

Now consider the problem N(III.1.3)

$$\text{Max } Z = 8x_1 + 23x_2$$

subject to

$$DX = d, \quad X \geq 0 \quad \begin{cases} -x_1 + x_2 + x_5 = 7 \\ -2x_1 + x_2 + x_6 = 4 \\ -5x_1 + 9x_2 + x_7 = 90 \\ 25x_1 + 27x_2 + x_8 = 675 \\ 21x_1 + 11x_2 + x_9 = 462 \\ x_2 + x_{10} = 15 \\ 5x_1 - 6x_2 + x_{11} = 60 \\ x_1 - 4x_2 + x_{12} = 8 \\ x_1, x_2, x_5, \ldots x_{12} \; 0 \end{cases} \qquad N(III.1.3)$$

The set of optimal solutions $X_1^{(3)}$ for N(III.1.3) is

$$\left[X_1^{(3)} = X_{11}^{(3)} = \left(\frac{54}{5}, 15, 0, 0, \frac{14}{5}, \frac{53}{5}, 9, 0, \frac{351}{5}, 0, 96, \frac{286}{5} \right) \right]$$

$$U_1^{(3)} = \frac{2157}{5}$$

$$X_1^{(3)} \cap S_1 = \emptyset$$

The value of the objective function at the extreme points adjacent to the element of $X_1^{(3)}$ is

$$\frac{2157}{5} - \frac{351 \times 359}{292} = \frac{100767}{292} = 345.04$$

and

$$\frac{2157}{5} - \frac{45 \times 8}{25} = 417$$

Therefore $V = 345.04 \quad \left(= \frac{100767}{292} \right)$

i.e. $V = \frac{100767}{292}$

Introduce the cut $8x_1 + 23x_2 \leq \frac{100767}{292}$ (called Deep Cut)

in N(III.1.3)

The problem so obtained is

$$\left.\begin{array}{l} \text{Max } Z = 8x_1 + 23x_2 \\ \text{subject to} \\ -x_1 + x_2 + x_5 = 7 \\ -2x_1 + x_2 + x_6 = 4 \\ -5x_1 + 9x_2 + x_7 = 90 \\ 25x_1 + 27x_2 + x_8 = 675 \\ 21x_1 + 11x_2 + x_9 = 462 \\ x_2 + x_{10} = 15 \\ 2x_1 - 6x_2 + x_{11} = 60 \\ x_1 - 4x_2 + x_{12} = 8 \end{array}\right| \quad N\ (III.1.4)$$

$$8x_1 + 23x_2 + x_{13} = \frac{100767}{292}$$

$$x_1, x_2, x_5, x_6, \ldots x_{13} \geq 0$$

The set of optimal solutions of N(III.1.4) is

$$X_1^{(4)} = \left[X_{11}^{(14)} = \left(\frac{53755}{9052}, \frac{117119}{9052}, 0, 0, 0, \frac{26599}{9052}, \frac{29384}{9052}, \frac{160402}{9052}, \frac{1764960}{9052}, \frac{18661}{9052}, \frac{977059}{9052}, \frac{487137}{9052} \right) \right.$$

$$\left. X_{12}^{(4)} = \left(\frac{5049}{292}, \frac{2625}{292}, 0, 0, \frac{4468}{292}, \frac{8641}{292}, \frac{27960}{292}, 0, 0, \frac{1755}{292}, \frac{8025}{292}, \frac{7787}{292}, \right) \right]$$

$$U_1^{(4)} = \frac{100767}{292}$$

The set $X_2^{(4)}$ of second best extreme point solution of N (III.1.4) is

$$X_2^{(4)} = \left[X_{21}^{(4)} = \left(\frac{3432}{181}, \frac{1050}{181}, 0, 0, \frac{3649}{181}, \frac{6538}{181}, \frac{24000}{181}, \frac{8025}{181}, 0, \frac{1665}{181}, 0, \frac{2216}{181}, \frac{3169875}{52852} \right) \right]$$

$$U_2^{(4)} = \frac{51606}{181}$$

$$X_2^{(4)} \cap S_1 = \emptyset$$

$\therefore$ Introduce $8x_1 + 23x_2 \leqslant \frac{51606}{181}$ in N(III.1.3), the problem so obtained is

$$\left.\begin{array}{l} \text{Max } Z = 8x_1 + 23x_2 \\ \text{subject to} \\ \quad -x_1 + x_2 + x_5 = 7 \\ \quad -2x_1 + x_2 + x_6 = 4 \\ \quad -5x_1 + 9x_2 + x_7 = 90 \\ \quad 25x_1 + 27x_2 + x_8 = 675 \\ \quad 21x_1 + 11x_2 + x_9 = 462 \\ \quad x_2 + x_{10} = 15 \\ \quad 5x_1 + 6x_2 + x_{11} = 60 \\ \quad x_1 - 4x_2 + x_{12} = 8 \\ \quad 8x_1 + 23x_2 + x_{14} = \frac{51606}{181} \\ \quad x_1, x_2, x_5, x_6, \ldots\ldots x_{12}, x_{14} \geqslant 0 \end{array}\right| \quad N\ (III.1.5)$$

The set of optimal extreme point solutions of N (III.1.5) is

$$X_1^{(5)} = X_{11}^{(5)} = \left(\frac{22465}{5611}, \frac{61742}{5611}, 0, 0, 0, \frac{5632}{5611}, \frac{61637}{5611}, \frac{1558766}{5611}, \frac{1441355}{5611}, \frac{22423}{5611}, \frac{594787}{5611}, \frac{269391}{5611} \right)$$

$$X_{12}^{(5)} = \left(\frac{3432}{181}, \frac{1050}{181}, 0, 0, \frac{3649}{181}, \frac{6538}{181}, \frac{24000}{181}, \frac{8025}{181}, 0, \frac{1665}{181}, 0, \frac{2216}{181}, 0 \right)$$

$$U_1^{(5)} = \frac{51606}{181}$$

Since $X_1^{(5)} \cap S_1 = \emptyset$, the set of second best extreme point solutions of N(III.1.5) is

$$X_2^{(5)} = \left[X_{21}^{(5)} = \left(3, 10, 0, 0, 0, \frac{5632}{181}, 15, 330, 289, 5, 105, 45 \right) \right]$$

$$U_2^{(5)} = 254$$

Now $X_2^{(5)} \cap S_1 = \emptyset$, introduce the cut $8x_1 + 23x_2 \leq 254$ in N(III.1.3) and the problem obtained is

$$\text{Max } Z = 8x_1 + 23x_2$$

subject to

$$\left.\begin{array}{rl} -x_1 + x_2 + x_5 & = 7 \\ -2x_1 + x_2 + x_6 & = 4 \\ -5x_1 + 4x_2 + x_7 & = 90 \\ 25x_1 + 27x_2 + x_8 & = 675 \\ 21x_1 + 11x_2 + x_9 & = 462 \\ x_2 + x_{10} & = 15 \\ 5x_1 - 6x_2 + x_{11} & = 60 \end{array}\right| \quad \text{N (III.1.6)}$$

$$x_1 - 4x_2 + x_{12} = 8$$

$$8x_1 + 23x_2 + x_{15} = 254$$

$$x_1, x_2, x_5, x_6, \ldots x_{12}, x_{15} \geq 0$$

The set $X_1^{(6)}$ of optimal extreme point solutions of N (III.1.6) is

$$X_1^{(6)} = X_{11}^{(6)} = (3,10,0,0,0,15,330,289,5,105,45,0)$$

$$X_{12}^{(6)} = \left[\left(\frac{2904}{163}, \frac{790}{163}, 0, 0, \frac{3255}{163}, \frac{5670}{163}, \frac{21265}{163}, \frac{16095}{163}, \frac{5632}{163}, \frac{1655}{163}, 0, \frac{1560}{163}, 0\right)\right]$$

$$U_1^{(6)} = 254$$

$X_1^{(6)} \cap S_1 = \emptyset$, the set of second best extreme point solutions of N(III.1.6) is

$$X_2^{(6)} = X_{21}^{(6)} = \left[\left(\frac{96}{7}, \frac{10}{7}, 0, 0, \frac{135}{7}, 30, \frac{985}{7}, \frac{2055}{7}, \frac{1108}{7}, \frac{95}{7}, 0, 0, \frac{780}{7}\right)\right]$$

$$U_2^{(6)} = \frac{998}{7}$$

Now $X_2^{(6)} \cap S_1 = \emptyset$, therefore introduce the cut $8x_1 + 23x_2 \leq \frac{998}{7}$ in N(III.1.3). The problem so obtained is

$$
\begin{aligned}
&\text{Max } Z = 8x_1 + 23x_2 \\
&\text{subject to} \\
&-x_1 + x_2 + x_5 = 7 \\
&-2x_1 + x_2 + x_6 = 4 \\
&-5x_1 + 9x_2 + x_7 = 90 \\
&25x_1 + 27x_2 + x_8 = 675 \\
&21x_1 + 11x_2 + x_9 = 462 \\
&x_2 + x_{10} = 15 \\
&5x_1 - 6x_2 + x_{11} = 60 \\
&x_1 - 4x_2 + x_{12} = 8 \\
&8x_1 + 23x_2 + x_{16} = \frac{998}{7} \\
&x_1, x_2, x_3, x_6, \ldots x_{12}, x_{16} \geq 0
\end{aligned}
\qquad N\ (\text{III}.1.7)
$$

The set of optimal extreme point solution is

$$
X_1^{(7)} = \Big[X_{11}^{(7)} = \left(\frac{177}{189}, \frac{370}{63}, 0, 0, \frac{135}{9}, 0, \frac{2635}{63}, \frac{31060}{63}, \frac{23797}{63}, \frac{575}{9}, \frac{815}{9}, \frac{275}{9}, 0 \right)
$$

$$
X_{12}^{(7)} = \left(\frac{96}{7}, \frac{10}{7}, 0, 0, 30, 30, \frac{1020}{7}, \frac{1915}{7}, \frac{1108}{7}, \frac{595}{7}, 0, 0, 0 \right) \Big]
$$

Now $X_1^{(7)} \cap S_1 = \emptyset$, the second best extreme point solution of N(III.1.7) is

$$X_2^{(7)} = \left[X_{21}^{(7)} = \left(0, 4, 0, 0, \frac{410}{27}, 0, 54, 567, 418, \frac{1381}{21}, 84, 24, \frac{354}{7} \right) \right]$$

$$U_2^{(7)} = 92$$

Now $X_2^{(7)} \cap S_1 \neq \emptyset$ (as it satisfies (a.1) and (a.2) both

$$x_1 = 0$$

$$x_2 = 4$$

is optimal extremepoint solution for N (III.1.1) and $Z = 92$ is the optimal value of objective function.

SECTION - 2

STRONG CUT CUTTING PLANE PROCEDURE FOR EPLPP

INTRODUCTION :

The procedure presented in this section is an improvement over the Deep Cut Procedure presented in Section I of this chapter for solving EPLPP in the sense that in this procedure, a much smaller subset of extreme points of $DX = d, X \geq 0$ is needed to be investigated to obtain the optimal solution of (II.1.1)

THEORETICAL DEVELOPMENT :

Consider the problem (II.1.2) viz.

$$\text{Max } Z = CX$$

subject to

$$FX = f$$

$$X \geq 0$$

The problem is assumed to be bounded because if it is not bounded it can always be converted into bounded one by introducing an additional constraint $CX \leq M$ as discussed in Chapter II. Apply simplex method to obtain $Y_1^{(2)}$, the set of optimal extreme point solutions of (II.1.2). If $Y_1^{(2)} = \emptyset$, then problem (II.1.1) and procedure is terminated. If $Y_1^{(2)} \neq \emptyset$ then determine $Y_1^{(2)} \cap S_1$. If $Y_1^{(2)} \cap S_1 \neq \emptyset$, then every element of $Y_1^{(2)} \cap S_1$ is optimal solution for (II.1.1). If $Y_1^{(2)} \cap S_1 = \emptyset$ then find the set $Y_2^{(2)}$ of second best extreme point solutions of (II.1.2). Let $V_1^{(2)}$ be value of objective function for the elements of $Y_1^{(2)}$ and $V_2^{(2)}$ the value of objective function for the elements of $Y_2^{(2)}$. If $Y_2^{(2)} = \emptyset$, then (II.1.1) has no solution and process is terminated. If $Y_2^{(2)} \neq \emptyset$, determine $Y_2^{(2)} \cap S_1$. If $Y_2^{(2)} \cap S_1 \neq \emptyset$, then every element of $Y_2^{(2)} \cap S_1$ is optimal extreme point solution of (II.1.1). Otherwise pick up the problem (II.2.1) viz.

$$\text{Max } Z = CX$$

subject to

$$DX = d$$

$$X \geq 0$$

In case (II.2.1) is unbounded a suitable constraint CX M as mentioned earlier makes it bounded. Find the set $X_1^{(1)}$ of optimal solution of (II.2.1) with $U_1^{(1)}$ the value of objective function corresponding to elements of $X_1^{(1)}$. If $X_1^{(1)} = \emptyset$, then problem (II.1.1) has no solution and the procedure is terminated. let $X_1^{(1)} \neq \emptyset$, since feasible region of (II.2.1) contains the feasible region of (II.1.2), therefore, $V_1^{(2)} \leq U_1^{(1)}$ (Equality holds in case $X_1^{(1)} \cap Y_1^{(2)} \neq \emptyset$). As $Y_1^{(2)} \cap S_1 = \emptyset$, $V_1^{(2)}$ is not optimal for (II.1.1) and $V_1^{(2)}$ $U_1^{(1)}$, therefore, $U_1^{(1)}$ is not optimal for (II.1.1). Hence $X_1^{(1)} \cap S_1 = \emptyset$. Now find all adjacent extreme points of the elements of the set of optimal extreme point solutions $X_1^{(1)}$, calculate the value of the objective function at these adjacent extreme points. Let R_i $V_2^{(2)}$ represent the value of objective function at those extreme points. Pick the value, say W_1, which is nearest to $V_2^{(2)}$ i.e. $W_1 = \text{Min} \left[R_i\right]$. Find the extreme points of $DX=d \geq o$ with value of objective function as W_1. Find the value of the objective function at the set of extreme points adjacent the extreme points with the value of objective function as W_1. Out of these values pick up the value, say W_2 ($\geq V_2^{(2)}$) which is nearest to $V_2^{(2)}$ and find the extreme points of $DX = d$, $X \geq o$ corresponding to W_2. Again, find the set of extreme points of $DX = d$, $X \geq o$ which are adjacent to the extreme points corresponding to W_2 and value of the objective function at these extreme points. Out of these pick up, say W_3 ($\geq V_2^{(2)}$), nearest to $V_2^{(2)}$ and find corresponding extreme points. This process is continued till a set of extreme points is obtained where value of objective function is W_k ($\geq V_2^{(2)}$) and at all the adjacent extreme points to these extreme points the value of the objective function is less than $V_2^{(2)}$. At this stage the cut

$$CX \leq W_k$$

is introduced in (II.2.1); $w_i < w_{i-1}$ $\forall_i$ and $w_i \geqslant V_2^{(2)}$ $I \leqslant i \leqslant k$. The problem so obtained is

$$\text{Max } Z = CX$$

subject to

$$DX = d \qquad \text{(III.2.3)}$$

$$CX \leqslant w_k$$

$$X \geqslant 0$$

The cut $CX \leqslant w_k$ is called 'Strong Cut'. Find out $X_1^{(3)}$ the set of optimal extreme point solutions of (III.2.3), the value of the objective function for the elements of $X_1^{(3)}$ will be, clearly, w_k. As $X_1^{(3)} \neq \emptyset$ and $w_k \geqslant V_2^{(2)}$ and $X_2^{(2)} \cap S_1 = \emptyset$, therefore, $X_1^{(3)} \cap S_1 = \emptyset$. The remaining extreme point solutions of (III.2.3) are determined in a systematic order by cutting plane technique (Given in Section 2 of Chapter II) till either an optimal extreme point solution of (II.1.1) is reached or an indication of no solution of (II.1.1) is obtained.

<u>EXAMPLE</u> :

In order to show the advantage of Strong Cut Procedure over the Deep Cut Procedure consider the problem N (III.1.1) solved in Section I of this Chapter.

SOLUTION :

From that problem

$$Y_2^{(2)} \cap S_1 = \emptyset$$

$$V_2^{(2)} = 128$$

Now consider the N(III.1.3)

The set of optimal solutions of N(III.1.3) is

$$X_1^{(3)} = X_{11}^{(3)} = \left(\frac{54}{5}, 15, 0, 0, \frac{14}{5}, 9, 0, \frac{351}{5}, 0, 96, \frac{286}{5} \right)$$

$$U_1^{(3)} = \frac{2157}{5}$$

$$X_1^{(3)} \cap S_1 = \emptyset$$

The value of the objective function at extreme point adjacent to $X_1^{(3)}$ are

$$R_1 = \frac{2157}{5} - \frac{45 \times 8}{25} = 417$$

$$R_2 = \frac{2157}{5} - \frac{351 \times 359}{292} = \frac{100767}{292}$$

$$V_1 = \frac{100767}{292} \quad (> V_2^{(2)})$$

The extreme point which has the value of the objective function as W_1 is

$$\left(\frac{5049}{292}, \frac{2625}{292}, 0, 0, \frac{1117}{73}, \frac{8641}{292}, \frac{6318}{73}, 0, 0, \frac{1725}{292}, \frac{8025}{292}, \frac{7787}{292} \right)$$

The value of objective function adjacent to this extreme point is

$$W_2 = \frac{51606}{181} \quad (> V_2^{(2)})$$

The extreme point which has the value of objective function as W_2 is

$$\left(\frac{3432}{181}, \frac{1050}{181}, 0, 0, \frac{3649}{181}, \frac{6538}{181}, \frac{22371}{181}, \frac{8025}{181}, 0, \frac{1665}{181}, 0, \frac{2216}{181} \right)$$

The value of objective function adjacent to this extreme point is

$$W_3 = \frac{998}{7} \quad (> V_2^{(2)})$$

The extreme point corresponding to W_3 is

$$\left(\frac{96}{7}, \frac{10}{7}, 0, 0, \frac{135}{7}, 30, \frac{957}{7}, \frac{2055}{7}, \frac{1108}{7} \right)$$

The value of the objective function adjacent to this extreme point is $W_4 = 64$ ($V_2^{(2)}$). Introduce the cut, called strong cut.

$$8x_1 + 23x_2 \leq \frac{998}{7} \quad \text{in} \quad N \text{ (III.1.3)}$$

The problem so obtained is

$$\text{Max} \quad Z = 8x_1 + 23x_2$$

subject to

$$\begin{aligned}
-x_1 + x_2 + x_5 &= 7 \\
-2x_1 + x_2 + x_6 &= 4 \\
-5x_1 + 9x_2 + x_7 &= 90 \\
25x_1 + 27x_2 + x_8 &= 675 \\
21x_1 + 11x_2 + x_9 &= 462 \\
x_2 + x_{10} &= 15 \\
5x_1 + 6x_2 + x_{11} &= 60 \\
x_1 + 4x_2 + x_{12} &= 8 \\
8x_1 + 23x_x + x_{13} &= \frac{998}{7} \\
x_1, x_2, x_5, x_6, \dots x_{13}, &\geq 0
\end{aligned}$$

The set of optimal extreme point solution of the problem is

$$X_1^{(4)} = \left[X_{11}^{(4)} = \left(\frac{177}{189}, \frac{370}{63}, 0, 0, 15, 0, \frac{2635}{63}, \frac{31060}{63}, \right.\right.$$

$$\left. \frac{23797}{63}, \frac{575}{9}, \frac{815}{9}, \frac{275}{9}, 0 \right)$$

$$X_{12}^{(4)} = \left(\frac{96}{7}, \frac{10}{7}, 0, 0, 30, 30, \frac{1020}{7}, \frac{1915}{7}, \frac{1108}{7}, \frac{595}{7}, 0, 0, 0 \right)$$

The second best extreme point solution of N(III.2.4) is

$$X_2^{(4)} = X_{21}^{(4)} = \left(0, 4, 0, 0, \frac{416}{27}, 0, 54, 567, 418, \frac{1381}{21}, 84, 24, \frac{354}{7} \right)$$

$$U_2^{(4)} = 92$$

Now $X_2^{(4)} \cap S_1 \neq \emptyset$, therefore

$$x_2 = 4$$

$$x_1 = 0$$

is the required optimal solution for the problem N (III.1.1) and $Z = 92$ is the optimal value of the objective function.

It may be noted that in Deep Cut procedure after apply Ceep Cut three cutting planes are introduced to get the optimal solution of N(III.1.1). Whereas in strong cut procedure only one cutting plane serves the purpose. Hence, Strong Cut procedure is in general more efficient than Deep Cut procedure.

SECTION 3

STRONG CUT (DEEP CUT) ENUMERATIVE TECHNIQUE FOR SOLVING EPLLP

INTRODUCTION:

In Section I and Section 2 of this Chapter cutting plane procedures are given. Here, in this section Strong Cut (Deep Cut) Enumerative Technique are presented.

THEORETICAL DEVELOPMENT :

After introducing Strong Cut (Deep Cut) $CX \leq W_k$ $(CX \leq V)$ in (II.1.2), the problem III.2.3 (III.1.3) is obtained. Find $X_1^{(3)}$, the set of optimal extreme point solutions of III.2.3 (III.1.3) and $U_1^{(3)}$ the value of objective function at the elements of $X_1^{(3)}$. Clearly $U_1^{(3)} = W_k$ ($U_1^{(3)} = V$). Also $X_1^{(3)} \cap S_1 = \emptyset$ let $B_1^{(3)}$ ($\neq \emptyset$) be the set of bases of the elements of $X_1^{(3)}$. Determine $E_1^{(3)}$, the set of bases which are adjacent to the elements of $B_1^{(3)}$ yielding the value of objective function less than $U_1^{(3)}$. Out of the elements of $E_1^{(3)}$ pick up the set $B_2^{(3)}$ of elements which yield the greatest value, say $U_2^{(3)}$, of objective function. This set generates $X_2^{(3)}$, the set of second best extreme point solutions of III.2.3 (III.1.3). If

$X_2^{(3)} = \emptyset$, then (II .1.1) has no solution. Otherwise determine $X_2^{(3)} \cap S_1$. If $X_2^{(3)} \cap S_1 \neq \emptyset$, then every element of $X_2^{(3)} \cap S_1$ is optimal for (II.1.1). In case $X_2^{(3)} \cap S_1 = \emptyset$ find $X_3^{(3)}$, the set of third best extreme point solutions of III.2.3 (III.1.3). Find out $E_2^{(3)}$, the set of all those bases which are adjacent to elements $B_2^{(3)}$ and yield the value of objective function less than $U_2^{(3)}$.

let $H_1^{(3)} = E_1^{(3)}$

$$H_2^{(3)} = \left[E_1^{(3)} \cup E_2^{(3)}\right] - B_2^{(3)}$$

Determine the set of elements, $B_3^{(3)}$, of $H_2^{(3)}$ which yield the greatest value, say $U_3^{(3)}$, of objective function.

This process is continued till for some (K+1) either $X_{(k+1}^{(3)} = \emptyset$ implying (III.1.1) has no solution or $X_{(k+1)}^{(3)} \neq \emptyset$ and $X_{(k+1)}^{(3)} \cap S_1 \neq \emptyset$ in which case every element of $X_{(k+1)}^{(3)} \cap S_1$ is optimal for (II.1.1) where $X_{(k+1)}^{(3)}$ is (k+1) st best extreme point of solution of III.2.3 (III.1.3) and generated by a subset of elements of

$$H_k^{(3)} = \left[\bigcup_{i=0}^{k-1} E_{i+1}^{(3)}\right] - \left[\bigcup_{i=1}^{k-1} B_{i+1}^{(3)}\right]$$

The indication $X^{(3)}_{(k+1)} = \emptyset$ is indicated by

$H_k^{(3)} = \emptyset$.

EXAMPLE :

The problem N(III.1.1) is solved here by strong cut Enumeration Technique.

SOLUTION :

After introducing the strong cut in N(III.2.2) we get N (III.2.4). The optimal solution to this problem is

$$X_1^{(4)} = \Big[X_{11}^{(4)} = \left(\frac{354}{378}, \frac{370}{63}, 0, 0, 15, 0, \frac{2635}{63}, \frac{31060}{63}, \frac{23797}{63}, \frac{575}{9}, \frac{815}{9}, \frac{275}{9}, 0 \right),$$

$$X_{12}^{(4)} = \left(\frac{96}{7}, \frac{10}{7}, 0, 0, 30, 30, \frac{1020}{7}, \frac{1915}{7}, \frac{1108}{7}, \frac{595}{7}, 0, 0, 0 \right) \Big]$$

The second best extreme point solution of N(III.2.4) is find out as follows :

$$B_1^{(4)} = \Big[B_{11}^{(4)} = (d_5, d_2, d_7, d_8, d_9, d_{10}, d_{11}, d_{12}, d_1)$$

$$B_{12}^{(4)} = (d_5, d_2, d_7, d_8, d_9, d_{10}, d_{11}, d_6, d_1) \Big]$$

$$E_1^{(4)} = \left\lfloor \begin{array}{l} E_{11}^{(4)} = (d_5, d_2, d_7, d_8, d_9, d_{10}, d_{11}, d_{12}, d_{13}) \\ E_{12}^{(4)} = d_5, d_{13}, d_7, d_8, d_9, d_{10}, d_{11}, d_6, d_1 \quad) \end{array} \right\rceil$$

The value of objective function for

$$E_{11}^{(4)} = \frac{998}{7} - \frac{354}{378} \times 54 = 92$$

$$E_{12}^{(4)} = \frac{998}{7} - \frac{10}{7} \times 55 = 64$$

$E_{11}^{(4)}$ generates $X_2^{(4)}$

$$X_2^{(4)} = X_{21}^{(4)} = \left(0, 4, 0, 0, \frac{416}{27}, 0, 5, 4, 567, 418, \frac{1381}{21}, 84, 24, \frac{354}{7} \right)$$

$$U_2^{(4)} = 92$$

Now $X_2^{(4)} \cap S_1 \neq \emptyset$ therefore,

$$x_1 = 0$$

$$x_2 = 4$$

is required optimal solution of problem N(III.1.1) and $Z = 92$ is the optimal value of objective function.

NOTE :

It may be noted that although for some problems the strong cut and Deep Cut procedures may coincide but in majority of cases, in the Strong Cut procedure, the investigation of a number or extreme points $DX=d, X \geq 0$ is avoided which are needed to be studied in Deep Cut procedure.

CHAPTER - IV

EXTREME POINT LINEAR FRACTIONAL FUNCTIONAL PROGRAMMING PROBLEM

SECTION 0

PRELIMINARIES :

In order to discuss Extreme Point linear Fractional Functional Programming Problem it is necessary to briefly survey the methods to solve a general linear Fractional Functional Programming Problem (LFPP). Certain results in this development are stated without proof details of which can be referred to paper references given. The general *LFPP* is

$$\left.\begin{array}{l} \text{Max} \quad \dfrac{CX + \alpha}{DX + \beta} \\ \text{subject to} \\ \qquad AX = b \\ \qquad X \geqslant 0 \end{array}\right\} \qquad (IV.0.1)$$

where X is nx1 matrix, C and D are 1xn matrics, A is mxn matrix and b is mx1 matrix, α, β are scales. It is assumed that the constraints of (IV.0.1) are regular i.e. the feasible region is non-empty and bounded. It is also assumed that denominator is non-zero for any feasible solution.

This problem has been studied by Charnes and Cooper (11) in which they have established that employing a linear transformation $Y = tX,\ t \quad 0$ reduces the problem (IV.0.1) to solving two equivalent linear programming problems viz.

$$\left.\begin{aligned} &\text{Max} \quad CY + \alpha t \\ &\text{subject to} \\ &\qquad AY - bt = 0 \\ &\qquad DY + \beta t = 0 \\ &\qquad\quad Y_1 t \geq 0 \end{aligned}\right\} \qquad (\text{IV}.0.2)$$

and

$$\left.\begin{aligned} &\text{Max} \quad - CY - \alpha t \\ &\text{subject to} \\ &\qquad AY - bt = 0 \\ &\qquad DY + \beta t = -1 \\ &\qquad\quad Y_1 t \geq 0 \end{aligned}\right\} \qquad (\text{IV}.0.3)$$

It is established that

(i) For every (Y, t) satisfying the constraints of (IV.0.2) and (IV.0.3) has $t \geq 0$

(ii) If $DX^{*} + \beta > 0$ for every optimal solution of (IV.0.1) and (Y^{*}, t^{*}) is an optimal solution of (IV.0.2) then Y^{*}/t^{*} is an optimal solution of (IV.0.1).

Similarly if $DX^* + \beta < 0$ for X^* an optimal solution of (IV.0.1) then replacing (c, α) and (D, β) by their negative the functional is unaltered and for the new (D, β) $DX^* + \beta > 0$ i.e. it becomes iquivalent to (IV.0.3).

Thus to solve (IV.0.1) it is sufficient to solve two ordinary linear programming problems viz. (IV.0.2) and (IV.0.3).

Another method to solve the problem (IV.0.1) is developed by K.Sawrup (33). This approach is developed on lines similar to solving a LPP by simplex method. It is assumed that

(i) Any m columns of A are linearly indopendant,

(ii) The denominator of the objective function is positive for feasible solutions.

Based on these assumptions it is established that optimal solution of (IV.0.1) is a basic feasible solution i.e. the optimum occurs at an extreme point of feasible region if $AX = b$, $X \geq 0$. Therefore, the procedure starts with a initial basic feasible solution, moves over the set of extreme points of the feasible region in such a way that in the absence of degenracy the value of objective function at each iteration is improved . Since the number of extreme points is finite and extreme point is repeated the procedure converges in a finite number of steps. The algorithm to solve (IV.0.1) is developed as follows:

Let X_B be initial basic feasible solution to the set of constraints (IV.0.1) which is obtained as in ordinary LPP. Let B be the corresponding basis, $B = (b_1, b_2, \ldots. b_m)$. Therefore, $X_B = B^{-1}b$, $X_B > 0$. Let C_B and D_B be the m component row vectors having their components as the coefficient associated with the basic variables in numerator and denominator of the objective function respectively. Corresponding to solution X_B, let

$$Z^{(1)} = C_B X_B + \alpha$$

$$Z^{(2)} = D_B X_B + \beta$$

Therefore, the value of the objective function of (IV.0.1) corresponding to solution X_B is

$$Z = \frac{Z^{(1)}}{Z^{(2)}}$$

It is required to determine a non-basic variable which when inserted in the basis B, according to the procedure of simplex method, should give an improved value of objective function.

Let a_j be the column of A not in B, then there exists scalers y_{ij} such that $a_j = B Y_j$ or $Y_j = B^{-1} a_j$. Also let $Z_j^{(1)} = C_B Y_j$, $Z_j^{(2)} = D_B Y_j$. Thus $Z_j^{(1)}$, $Z_j^{(2)}$, Y_j are know for every column a_j of A not in B. Suppose column b_r of B is replaced by a_j of A not in B by means of simplex method for LPP to obtain a new basic feasible solution $\hat{X}_B$ where

$$\hat{X}_{Bi} = X_{BI} - X_{Br} \frac{y_{ij}}{y_{rj}} \quad i \neq r$$

$$\hat{X}_{Br} = \frac{X_{Br}}{y_{rj}} = \theta$$

let the new value of the objective function be

$$Z = \frac{Z^1}{Z^2}$$

where

$$\hat{Z}^1 = Z^{(1)} - \theta \, (Z_j^{(1)} - C_j)$$

$$\hat{Z}^2 = Z^{(2)} - \theta \, (Z_j(2) - D_j)$$

Now the value of objective function improves if

$\hat{Z} > Z$ which implies

$$\theta \quad Z^{(1)} (Z^{(2)} - D_j) - Z^{(2)} (Z^{(1)} - C_j) > 0$$

In the absence of degeneracy the new basic feasible solution improves the value of the objective function iff $\Delta_j > 0$ where $\Delta_j = \left[Z^{(1)} (Z^{(2)} - D_j) - Z^{(2)} (Z^{(1)} - C_j) \right]$. Also for the new basic solution to be feasible there must be at least one $y_{ij} > 0$ $i = 1, 2, \ldots n$. Thus any column a_j of A not in B if entered in the basis B gives an improved value of objective function if:

(i) There is at least one $y_{ij} > 0$, $i = 1, 2, \ldots n$ with $X_{Bi} > 0$.

(ii) For the column a_j, $\Delta_j > 0$ and $\Delta_j = \operatorname{Max}_k \Delta_k$, $\Delta_k > 0$ so that the improvement in the value of objective function is rapid.

It may also be noted that for every column in the basis $\Delta_j = 0$.

The procedure will terminate i.e. optimality will be achieved when $\Delta_j \leq 0$ $\forall$ j.

It is also established in paper (10) that for any a_j column of A not in B there is at least one $y_{ij} > 0$, $i = 1, 2, \ldots m$ because in the contrary case when all $y_{ij} \leq 0$, the solution set of (IV.0.1) becomes unbounded which is a contradiction. The proof of this is exactly similar to that for LPP.

SECTION - I
CUTTING PLACE PROCEDURE - I FOR SOLVING

This section is devoted to procedure of finding the optimal extreme point solution to Extreme Point linear Fractional Functional Programming Problem (EPLFPP)

INTRODUCTION :

An EPLFPP seeks to optimize an objective function, which is ratio of two linear functions, subject to linear constraints. The most general EPLFPP is

$$\text{Max} \quad L(X) = \frac{CX+\alpha}{DX+\beta}$$

subject to

$$AX = b$$

$$X \geq 0$$

and X is extreme point of

$$RX = t$$

$$X \geq 0$$

where A is mxn matrix, R is pXn matrix t is px1 matrix, b is mx1 matrix, X is nx1 matrix C, D are IXn matrims and α_1 β are scalers. The procedure discussed here moves over the extreme points of the convex polyhydron $AX = b$ $RX = t$, $X \geq 0$ till an extreme point of $RX=t$, $X \geq 0$ is obtained.

THEORETICAL DEVELOPMENT :

let

$J = [r_j : r_j \neq 0$ where r_j is j^{th} column of $R]$

$J(X) = [r_j \in J : x_j \neq 0$ where $X = (x_1, x_2, \ldots x_n)]$

$S_1 = [X : AX = b$, X is an extreme point of $RX = t,\ X \geq 0]$

$S_2 = [X : X$ is an extreme pointof $FX = f,\ X \geq 0]$

In order to solve (IV.1.1) start with the problem

$$\left.\begin{array}{l} \text{Max } \; I(X) \equiv \dfrac{CX + \alpha}{DX + \beta} \\ \text{subject to} \\ \quad FX = f \\ \quad X \quad 0 \end{array}\right| \quad (IV.1.2)$$

Where $F = \begin{bmatrix} A \\ R \end{bmatrix}$, $f = \begin{bmatrix} b \\ t \end{bmatrix}$

Problem (IV.1.1) is always bounded because solution of (IV.1.1) is an extreme point of $RX = t$, $X \geq 0$ and extreme points of $RX=t$, $X \geq 0$ are finite problem (IV.1.2) may be bounded or unbounded. If (IV.1.2) is unbounded it can always be converted into bounded one by introducing the constraint. In $X \leq M$, M is positive, finite, large number chosen in such a way that all the extreme points of (IV.1.2) are taken into account, in problem (IV.1.2). Henceforth, assume that (IV.1.2) is bounded.

THEOREM : $S_1 \subseteq S_2$

PROOF : Refer to Chapter II

Let X_B be initial basic feasible solution for the constraints of (IV.1.2) and $B=(f_1, f_2, \ldots f_{m+p})$ be basis for X_B therefore $X_B = B^{-1}f$, X_B 0. Let C_B and D_B be the (m+p) component row vectors having their components as the coefficient associated with the basic variables in numerator and denominator the objective function respectively.

Corresponding to solution

X_B let

$$z^{(1)} = C_B X_B + \alpha$$

$$z^{(2)} = D_B X_B + \beta$$

The value of objective function of (IV.1.1) corresponding to solution X_B is

$$z = \frac{z^{(1)}}{z^{(2)}}$$

Also for any column f_j of f not in B $Y_j = B^{-1} f_j$ and $Z_j^{(1)} = C_B Y_j$, $Z_j^{(2)} = D_B Y_j$. Find the value of the objective function at an extreme point adjacent to the extreme point corresponding to X_B i.e. a new basic feasible solution by changing only one column of B.

Let the new basic feasible solution be $\hat{X}_B$ and the corresponding basis be $\hat{B} = (\hat{f}_1, \hat{f}_2 \ldots\ldots \hat{f}_{m+p})$ where $\hat{f}_i = f_i,\ i \neq r$; $\hat{f}_r = f_j$. Now $\hat{X}_B = \hat{B}_f^{-1}$ and

$$\hat{X}_{Bi} = X_{Bi} - X_{Br} \frac{y_{ij}}{y_{rj}} \quad i \neq r$$

$$\hat{X}_{Br} = \frac{X_{Br}}{Y_{rj}} = \theta_j \quad \text{(say)}$$

The value of objective function for this extreme point solution is

$$\hat{Z} = \frac{\hat{Z}^1}{\hat{Z}^2}$$

where $$\hat{Z} = Z^{(1)} + \theta_j (c_j - z_j^{(1)})$$

$$\hat{Z} = Z^{(2)} + \theta_j (D_j - Z_j^{(2)})$$

The net change in the value of objective function is

$$\bar{Z}_j = \bar{Z} - Z$$

$$= \frac{Z^{(1)} + \theta_j \left(c_j - z_j^{(1)} \right)}{Z^{(2)} + \theta_j \left(D_j - Z_j^{(2)} \right)} - \frac{Z^{(1)}}{Z^{(2)}}$$

$$\bar{Z}_j = \frac{\theta_j \left(Z^{(1)} (Z_j^{(2)} - D_j) - z^{(2)} (z_j^{(1)} - c_j) \right)}{z^{(2)} \left[z^{(2)} - \theta_j (z_j^{(2)} - D_j) \right]}$$

$$= \frac{\theta_j \Delta_j}{z^{(2)} \left[z^{(2)} - \theta_j (z_j^{(2)} - D_j) \right]}$$

Therefore, for a EPLFPP the net change in the value of objective function of (IV.1.2) while moving from one extreme point to another (adjacent) extreme point is

$$\bar{Z}_j \qquad \frac{\theta_j \Delta_j}{z^{(2)} \left[z^{(2)} - \theta_j (z_j^{(2)} - D_j) \right]}$$

Now apply simplex method to find the set $X_1^{(2)}$ of optimal extreme point solutions of (IV.1.2) and $U_1^{(2)}$ the value of objective function for the elements of $X_1^{(2)}$.

Let the rank of R be p and rank of f be $(m+p)$ non-zero components and if $X \in S_1$ then X has at most p non-zero components. If $X \in S_2$ and $J(X) > p$ then $X \in S_1$ and if $X \in S_2$ and $|J(X)| \leq p$, the linear independence of elements of $J(X)$ implies $X \in S_1$.

Now if $X_1^{(2)} = \emptyset$, then problem (IV.1.1) has no solution and procedure is terminated. Otherwise determine $X_1^{(2)} \cap S_1$. If $X_1^{(2)} \cap S_1 \neq \emptyset$, then every element of $X_1^{(2)} \cap S_1$ is optimal for (IV.1.1). If $X_1^{(2)} \cap S_1 = \emptyset$, then, find $X_2^{(2)}$ the set of second best extreme point solution of (IV.1.2) as follows:

Determine

$$H(B) = \left[j : \Delta_j < 0 \right]$$

$$\theta_j = \mathrm{Min} \left[\frac{X_{Bi}}{y_{ij}},\ y_{ij} > 0 \right],\ j \in H(B)$$

$$_B = \underset{j \in H(B)}{\mathrm{Min}} \left[- \Delta_j\ ,\ \theta_j > 0 \right]$$

and

$$\delta = \mathrm{Min}\ /_B : B \text{ is basis for an element of } X_1^{(2)}$$

The δ gives a basis and column to be removed and entered in the basis. The basis so obtained generate the set $X_2^{(2)}$ of second best extreme point solution of (IV.1.2). Let $X_2^{(2)} = X_{21}^{(2)}, X_{22}^{(2)}, \ldots\ldots\ X_{2k_2}^{(2)}$ and $U_2^{(2)}$ be value of objective function for the elements of $X_2^{(2)}$. If $X_2^{(2)} = \emptyset$ then (IV.1.1) has no solution and procedure is terminated. Otherwise determine

$X_2^{(2)} \cap S_1$. If $X_2^{(2)} \cap S_1 \neq \emptyset$, then every element of $X_2^{(2)} \cap S_1$ is optimal for (IV.1.1). If $X_2^{(2)} \cap S_1 = \emptyset$, then introduce the cut $L(X) \leq U_2^{(2)}$ in (IV.1.2).

The problem so obtained is

$$\left.\begin{aligned} &\text{Max} \quad L(X) = \frac{CX + \alpha}{DX + \beta} \\ &\text{subject to} \\ &\qquad FX = f \\ &\qquad L(X) \leq U_2^{(2)} \\ &\qquad X \geq 0 \end{aligned}\right\} \qquad \text{(IV.1.3)}$$

Find the set $X_1^{(3)}$ of optimal extreme point solution of (IV.1.3). Now $X_1^{(3)} \neq \emptyset$ and $X_1^{(3)} \cap S_1 = \emptyset$, determine $X_2^{(3)}$, the set of second best extreme point solutions of (IV.1.3). If $X_2^{(3)} = \emptyset$, then problem (IV.1.1) has no solution. Otherwise, determine

$X_2^{(3)} \cap S_1$. If $X_2^{(3)} \cap S_1 \neq \emptyset$, then every element of $X_2^{(3)} \cap S_1$ is optimal for (IV.1.1). If $X_2^{(2)} \cap S_1 = \emptyset$ introduce the cut $L(X) \leq U_2^{(3)}$ in (IV.1.2), where $U_2^{(3)}$ is the value of objective function for the elements of $X_2^{(3)}$. The problem so obtained is

$$\begin{aligned} &\text{Max} \quad L(X) = \frac{CX + \alpha}{DX + \beta} \\ &\text{subject to} \\ &\qquad FX = f \\ &\qquad L(X) \leq U_2^{(3)} \\ &\qquad X \geq 0 \end{aligned} \qquad \text{(IV.1.4)}$$

Find $X_2^{(4)}$, the set of second best extreme point solutions of (IV.1.4).

This process is continued till, for some k, either $X_2^{(k)} = \emptyset$ which implies that (IV.1.1) has no solution or $X_2^{(k)} \neq \emptyset$ and $X_2^{(k)} \cap S_1 \neq \emptyset$ which implies that every element of $X_2^{(k)} \cap S_1$ is the set of second best extreme point solutions of (IV.1.k)

$$\begin{aligned} &\text{Max } L(X) = \frac{CX + \alpha}{DX + \beta} \\ &\text{subject to} \\ &\qquad FX = f \\ &\qquad L(X) \leq U_2^{(k-1)} \\ &\qquad X \geq 0 \end{aligned} \qquad \text{(II.1.k)}$$

where $U_2^{(k-1)}$ is the value of objective function for the elements of $X_2^{(k-1)}$

Note that $X_2^{(k)} = X_k^{(2)}$, the set of k^{th} best extreme point solutions of (IV.1.2)

The procedure given converges in finite number of steps as $S_1 \subseteq S_2$ and S_2 is finite and no extreme point is repeated.

The cuts $L(X) \leq U_2^{(i)}$ are angular cuts passing through the intersection of $CX+\alpha=0$ and $DX+\beta=0$ and a cut at any stage makes the previous cuts redundant.

<u>EXAMPLE</u> :

$$\text{Max} \quad L(X) = \frac{x_1 + 6x_2}{x_1 + 6}$$

subject to

$$-x_1 + 2x_2 \leq 2$$
$$3x_1 + 7x_2 \leq 21$$
$$x_1, \; x_2 \geq 0$$

and (x_1, x_2) is extreme point of

$$-x_1 + x_2 \leq 1$$
$$x_1 + x_2 = 6$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

N (IV.1.1)

SOLUTION :

For solving N(IV.1.1) start with the problem

$$\text{Max } U(X) = \frac{x_1 + 6x_2}{x_1 + 6}$$

subject to

$$FX = f, \quad X \geq 0 \qquad \left.\begin{aligned} -x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + 7x_2 + x_4 &= 21 \\ -x_1 + x_2 + x_5 &= 1 \\ x_1 + x_2 + x_6 &= 6 \\ x_1 - x_2 + x_7 &= 2 \\ x_1, x_2, \ldots\ldots\ldots x_7 &\; 0 \end{aligned}\right\} \quad N\ (IV.1.2)$$

The set of optimal solutions of N (IV.1.2) is

$$X_1^{(2)} = X_{11}^{(2)} = \left(\frac{28}{13}, \frac{27}{13}, 0, 0, \frac{14}{13}, \frac{23}{13}, \frac{25}{13} \right)$$

and

$$U_1^{(2)} = \frac{95}{33}$$

OPTIMAL TABLE FOR $X_{11}^{(2)}$

C_B	D_B	Variables of the basis		f_1	f_2	f_3	f_4	f_5	f_6	f_7
			C_j	1	6	0	0	0	0	0
			D_j	1	0	0	0	0	0	0
			X_B							
6	0	f_2	27/13	0	1	1/13	0	0	0	0
1	1	f_1	28/13	1	0	7/13	2/13	0	0	0
0	0	f_5	14/13	0	0	-10/13	1/13	1	0	0
0	0	f_6	23/13	0	0	4/13	-3/13	0	1	0
0	0	f_7	25/13	0	0	10/13	-1/13	0	0	1
$Z^{(1)} = \frac{190}{13}$		$Z_j^{(1)} - C_j$		0	0	$\frac{11}{13}$	$\frac{8}{13}$	0 8	0	0
$Z^{(2)} = \frac{106}{13}$		$Z_j^{(2)} - D_j$		0	0	$-\frac{7}{13}$	$\frac{2}{13}$	0	0	0
$Z = \frac{95}{53}$		γ_j		0	0	$-\frac{192}{13}$	$-\frac{36}{13}$	0	0	0

$J = \{ r_1, r_2, r_5, r_6, r_7 \}$

$p = 3$

$|J(X_{11}^{(2)})| = 573$

$X_1^{(2)} \cap S_1 = \emptyset$

Now the set of second best extreme point solutions of N(IV.1.2) is determined as follows:

Let B be basis for $X_{11}^{(2)}$ an element of $X_1^{(2)}$.

$H(B) = \{3, 4\}$

$\theta_3 = \frac{5}{2}$

$\theta_4 = 14$

$\Delta_3 = -\frac{420}{1007} = -.476$

$\Delta_4 = -\frac{52}{53} = -.98$

$_B = \text{Min} \{.476, .98\} = .476$

$\delta = .476$

Replacing f_7 by f_3 in B, a basis which generates an element of $X_2^{(2)}$, the set of second best extreme point solution of N(IV.1.2) is obtained.

$X_2^{(2)} = \{X_{21}^{(2)} = (\frac{7}{2}, \frac{3}{2}, \frac{5}{2}, 0, 3, 1, 0)\}$

$U_2^{(2)} = \frac{25}{19}$

$|J(X_{21}^{(2)})| = 4 > 3$

$X_2^{(2)} \cap S_1 = \emptyset$

Introduce the cut $\frac{x_1 + 6x_2}{x_1 + 6} \leq \frac{25}{19}$ in N(IV.1.2)

i.e. $-x_1 + 19x_2 \leq 25$

The problem so obtained is

$$\left.\begin{aligned}
&\text{Max} \quad L(X) = \frac{x_1 + 6x_2}{x_1 + 6} \\
&\text{subject to} \\
&\quad -x_1 + 2x_2 + x_3 = 2 \\
&\quad -3x_1 + 7x_2 + x_4 = 21 \\
&\quad -x_1 + x_2 + x_5 = 1 \\
&\quad x_1 + x_2 + x_6 = 6 \\
&\quad x_1 - x_2 + x_7 = 2 \\
&\quad -x_1 + 19x_2 + x_8 = 25 \\
&\quad x_1, x_2, \ldots\ldots x_8 \geq 0
\end{aligned}\right| \quad N\ (IV.1.3)$$

The set of optimal solutions of N(IV.1.3) is

$$X_1^{(3)} = \Big[X_{11}^{(3)} = \left(\frac{12}{17}, \frac{23}{17}, 0, \frac{160}{17}, \frac{6}{17}, \frac{67}{17}, \frac{45}{17}, 0 \right)$$

$$X_{12}^{(3)} = \left(\frac{7}{2}, \frac{3}{2}, \frac{5}{2}, 0, 3, 1, 0, 0 \right) \Big]$$

The set of second best extreme point solutions of N (IV.1.3) is

$X_2^{(3)} = \lfloor X_{21}^{(3)} = (0, 1, 0, 14, 0, 5, 3, 6) \rfloor$

$U_2^{(3)} = 1$

$|J (X_{21}^{(3)})| = 3 = p$

$\therefore X_2^{(3)} \cap S_1 \neq \emptyset$

Hence $x_1 = 0, x_2 = 1$ is required solution of N(IV.1.1) and optimal value of objective function is $L(X) = 1$.

SECTION - 2
CUTTING PLANE PROCEDURE II FOR SOLVING EPLPP

INTRODUCTION :

The procedure for solving EPLFPP given in Section I of this Chapter is computationally lengthy because

(a) At each iteration the linear independence of a subset of columns of D is to be checked.

(b) Alternate optimal solutions whuch are not extreme points of $RX = t, X \geq 0$ are to be studied.

The procedure discussed here removes the difficulty. The procedure discussed in this section moves from one extreme point to another extreme point solution of $RX=t, X \geq 0$ till the feasibility in $AX=b$ is satisfied.

<u>THEORETICAL DEVELOPMENT</u> :

let S = X : X is an extreme point of RX = t, X ≥ 0

Clearly, S is finite.

Problem (IV.1.1) can be restated as

$$\begin{aligned} &\text{Max } L(X) \equiv \frac{CX + \alpha}{DX + \beta} \\ &X \in S \\ &\text{subject to} \\ &\qquad AX = b \\ &\qquad X \geq 0 \end{aligned} \qquad \text{(IV.2.1)}$$

Then, as discussed in Section 2, Chapter, any iterative procedure which does the following solve the problem (IV.1.1)

<u>STEP 1</u> :

At i^{th} iteration i^{th} best extreme point solutions of

$$\text{Max } L(X) \equiv \frac{CX + \alpha}{DX + \beta}$$

$$X \in S$$

are found.

<u>STEP 2</u> :

The elements found in STEP 1 are tested for the feasibility in AX = b

STEP 3:

If test in STEP 2 is positive then procedure is terminated otherwise (i+1)st iteration is performed.

In order to solve (IV.1.1) start with the problem

$$\left.\begin{aligned} &\text{Max} \quad L(X) = \frac{CX + \alpha}{DX + \beta} \\ &\quad \text{subject to} \\ &\qquad RX = t \\ &\qquad X \geq 0 \end{aligned}\right\} \qquad (IV.2.2)$$

The problem (IV.1.1) is always bounded but (IV.2.2) may be bounded or unbounded. If (IV.2.2) is unbounded then the inclusion of a constraint. In $X \leq M$, where M is positive, finite, large number so that no extreme point of (IV.2.2) is excluded.

Apply simplex method to find $X_1^{(2)}$, the set of optimal extreme point solutions of (IV.2.2). If $X_1^{(2)} = \emptyset$ then (IV.1.1) has no solution and procedure is terminated. Otherwise determine $X_1^{(2)} \cap S_1$. If $X_1^{(2)} \cap S_1 \neq \emptyset$, then every element of $X_1^{(2)} \cap S_1$ is optimal for (IV.1.1). If $X_1^{(2)} \cap S_1 = \emptyset$, then find $X_2^{(2)}$, the set of second best extreme point solutions of (IV.2.2). If $X_2^{(2)} = \emptyset$, then (IV.1.1) has no solution and procedure is terminated. Otherwise determine $X_2^{(2)} \cap S_1$. If $X_2^{(2)} \cap S_1 \neq \emptyset$, then every element of $X_2^{(2)} \cap S_1$ is optimal for (IV.1.1). If $X_2^{(2)} \cap S_1 = \emptyset$, then remaining extreme point solutions of (IV.2.2) are studied in a systematic order by cutting plane technique (as explained in Section I of the present chapter) till either an indication of no solution of (IV.1.1) is obtained or an extreme point of (IV.2.2) is reached which satisfies feasibility in $AX = b$.

EXAMPLE :

$$\left.\begin{array}{l}
\text{Max} \quad L(X) = \dfrac{x_1 + 6x_2}{x_1 + 4x_2 + 2} \\
\text{subject to} \\
\quad -2x_1 + x_2 \leq 1 \qquad (a.1) \\
\quad 2x_1 + 10x_2 \leq 19 \qquad (a.2) \\
\quad x_1, x_2, \geq 0 \\
\text{and } (x_1, x_2) \text{ is extreme point of} \\
\quad -x_1 + x_2 \leq 2 \\
\quad x_1 + 2x_2 \leq 8 \\
\quad x_1, x_2, \geq 0
\end{array}\right\} \quad N(IV.2.1)$$

<u>SOLUTION</u> :

In order to solve N(Iv.2.1) consider the following problem

$$\text{Max} \quad L(X) = \frac{x_1 + 6x_2}{x_1 + 4x_2 + 2}$$

subject to

$$RX = t \quad \left\{ \begin{array}{l} -x_1 + x_2\ x_3 = 2 \\ x_1 + 2x_2 + x_4 = 8 \\ x_1, x_2, x_3, x_4 \geqslant 0 \end{array} \right. \qquad N\,(IV.2.2)$$

$$X \quad 0$$

The optimal solution of N(IV.2.2) is

$$X_1^{(2)} = \left[X_{11}^{(2)} \left(\frac{4}{3}, \frac{10}{3}, 0, 0 \right) \right]$$

$$U_1^{(2)} = \frac{32}{25}$$

Now $X_1^{(2)} \cap S_1 = \emptyset$, $X_2^{(2)}$, the set of second best extreme point solutions of N(IV.2.2) is

$$X_2^{(2)} = \left[X_{21}^{(2)} = (0, 2, 0, 4) \right]$$

$$U_2^{(2)} = \frac{6}{5}$$

Now $X_2^{(2)} \cap S_1 = \emptyset$, the cut $\frac{x_1 + 6x_2}{x_1 + 4x_2 + 2} \leq \frac{6}{5}$

i.e. $- x_1 + 6x_2 \quad 12$ is introduced in N(IV.2.2).

The problem so obtained is

$$\left.\begin{aligned} &\text{Max} \quad L(X) = \frac{x_1 + 6x_2}{x_1 + 4x_2 + 2} \\ &\text{subject to} \\ &-x_1 + x_2 + x_3 = 2 \\ &x_1 + 2x_2 + x_4 = 8 \\ &-x_1 + 6x_2 + x_5 = 12 \\ &x_1, x_2, \ldots x_5 \geq 0 \end{aligned}\right\} \quad N(IV.2.3)$$

The set of optimal solutions of N(IV.2.3) is

$$X_1^{(3)} = \left[\begin{aligned} X_{11}^{(3)} &= (0, 2, 0, 4) \\ X_{12}^{(3)} &= (3, \tfrac{5}{2}, 0, \tfrac{5}{2}) \end{aligned} \right] .$$

The set $X_2^{(3)}$ of second best extreme point solution of N(IV.2.3) is

$$X_2^{(3)} = \left[X_{21}^{(3)} = (8, 0, 0, 10, 2, 0) \right]$$

Now $X_2^{(3)} \cap S_1 \neq \emptyset$ as $X_{21}^{(3)}$ satisfies both (a.1) and (a.2). Hence $x_1 = 8$, $x_2 = 0$ is the required solution for N(IV.1.1) and optimal value of objective function is $L(X) = \frac{4}{5}$.

SECTION - 3

ENUMERATION TECHNIQUE FOR SOLVING EPLPP

INTRODUCTION :

In cutting plane procedures given in Section I and Section 2 of this chapter some unwanted alternate extreme point solutions have to be studied. To overcome this difficulty enumerative procedures are presented here.

In this section enumerative procedures for solving EPLFPP starting with the problems (IV.1.2) and (IV.2.2) are presented.

THEORETICAL DEVELOPMENT :

(a) Enumerative Technique for solving EPLFPP starting with the problem (IV.1.2) ;

Start with the problem (IV.1.2) viz.

$$\text{Max} \quad L(X) \equiv \frac{CX + \alpha}{DX + \beta}$$

subject to

$$FX = f$$

$$X \geq 0$$

Where $F = \begin{bmatrix} A \\ R \end{bmatrix} \quad f = \begin{bmatrix} b \\ t \end{bmatrix}$

Assume that :

(i) Problem (IV.1.2) is bounded

(ii) The set $S^* = \{X : FX = f, X \geq 0\}$ is non-empty and bounded.

(iii) $DX + \beta > 0 \;\forall \in S^*$

Since S^* is non-empty and bounded one of following exist :

(R 1) (IV.1.2) has a solution (IV.1.1) has a solution

(R 2) (IV.1.2) has a solution (IV.1.1) has no solution

(Both the cases are treated simultaneously)

Find the set $X_1^{(2)}$ of optimal extreme point solutions of (IV.1.2). Determine $X_1^{(2)} \cap S_1$. If $X_1 \cap S_1 \neq \emptyset$ then every element of $X_1^{(2)} \cap S_1$ is optimal solution for (IV.1.1). Otherwise find $X_2^{(2)}$, the set of second best extreme point solutions of (IV.1.2) as follows:

Let $B_1^{(2)}$ be set of bases for the elements of $X_1^{(2)}$ and $U_1^{(2)}$ be the value of objectivefunction at on element of $X_1^{(2)}$. Find $E_1^{(2)}$, the set of all those bases which are adjacent to the elements of $B_1^{(2)}$ and yield the value of objective function less than $U_1^{(2)}$. Out of these values of objective function pick up the greatest, say $U_2^{(2)}$. The subset $B_2^{(2)}$ of $E_1^{(2)}$, the elements of which yield the value of objective function as $U_2^{(2)}$, generate the set $X_2^{(2)}$. If $X_2^{(2)} = \emptyset$ then (IV.1.1) has no solution and the procedure is terminated. Otherwise determine $X_2^{(2)} \cap S_1$. If $X_2^{(2)} \cap S_1 \neq \emptyset$ then every element of $X_2^{(2)} \cap S_1$ is optimal for (IV.1.1). If $X_2^{(2)} \cap S_1 = \emptyset$, then find $X_3^{(2)}$, the set of third best extreme point solutions of (IV.1.2) as follows:

$$\text{let } H_1^{(2)} = E_1^{(2)}$$

$$H_2^{(2)} = (E_1^{(2)} \cup E_2^{(2)}) - B_2^{(2)}$$

where $E_2^{(2)}$ is the set of all those bases which are adjacent to the elements of $B_2^{(2)}$ and yield the value of objective function less than $U_2^{(2)}$. A subset $B_3^{(2)}$ of elements of $H_2^{(2)}$ which yield the greatest value, say $U_3^{(2)}$, of the objective function generate the set $X_3^{(2)}$. If $X_3^{(2)} = \emptyset$, then (IV.1.1) has no solution and procedure is terminated. If $X_3^{(2)} \neq \emptyset$ then determine $X_3^{(2)} \cap S_1$. If $X_3^{(2)} \cap S_1 \neq \emptyset$, then every element of $X_3^{(2)} \cap S_1$ is optimal for (IV.1.1). Otherwise find $X_4^{(2)}$, the set of fourth best extreme point solutions of (IV.1.2).

This process is continued till, for some k, either $X_k^{(2)} \neq \emptyset$ and $X_k^{(2)} \cap S_1 \neq \emptyset$ which implies that every element of $X_k^{(2)} \cap S_1$ is optimal for (IV.1.1) or $X_k^{(2)} = \emptyset$ indicated by $H_{k-1}^{(2)} = \emptyset$, implying (IV.1.1) has no solution.

(b) Enumerative Technique for solving EPLFPP starting with problem (IV.2.2).

Start with the problem (IV.2.2) viz.

$$\text{Max} \quad L(X) \equiv \frac{CX + \alpha}{DX + \beta}$$

subject to

$$RX = t$$

$$X \geq 0$$

Assume that :

(i) $\bar{S}^x = X : RX = t, \; X \geq 0$ is finite and bounded

(ii) $DX + \beta > 0 \;\; \forall X \in \bar{S}^*$

(iii) Problem (IV.2.2) is bounded

The extreme points of (IV.2.2) are ranked by enumerative technique as explained in part (a) of this section till an extreme point solution of $RX=t$, $x \geq 0$ which satisfies feasibility in $AX = b$ is achieved.

This procedure is preferred because in it testing the extreme points of RX≤t, $x \geq 0$ is much simple. Also in this procedure the basis are of smaller size.

EXAMPLE :

Here problem (IB.2.1) is solved by enumerative technique given in (b)

SOLUTION :

Start with the problem H(IV.2.2).

Now
$$X_1^{(2)} = \left[X_{11}^{(2)} = \left(\frac{4}{3}, \frac{10}{3}, 0, 0 \right) \right]$$

$$U_1^{(3)} = \frac{32}{25}$$

$X_1^{(2)} \cap S_1 = \emptyset$, therefore $X_2^{(2)}$, the set of second best extreme point solution is determined as follows:

$$B_1^{(2)} = \left[B_{11}^{(2)} = (r_2, r_1) \right]$$

$$E_1^{(2)} = \left[E_{11}^{(2)} = (r_3, r_1), E_{12}^{(2)} = (r_2, r_4) \right]$$

The value of objective function for

$$E_{11}^{(2)} = \frac{4}{5} = 8$$

$$E_{12}^{(2)} = \frac{18}{15} = 1.2$$

$E_{12}^{(2)}$ generates an element of the set $X_2^{(2)}$ and

$$X_2^{(2)} = X_{21}^{(2)} = (0, 2, 0, 4)$$

$X_2^{(2)} \cap S_1 = \emptyset$, therefore, $X_3^{(2)}$, the set of third best extreme point solution of N(IV.2.2) is found.

$$B_2^{(2)} = \left[B_{21}^{(2)} = (r_2, r_4)\right]$$

$$E_2^{(2)} = \left[E_{21}^{(2)} = (r_3, r_4)\right]$$

$$H_2^{(2)} = \left[E_1^{(2)} \cup E_2^{(2)} - B_2^{(2)}\right]$$

$$= \left[H_{21}^{(2)} = (r_3, r_1), H_{22}^{(2)} = (r_3, r_4)\right]$$

The value of objective function for

$$H_{21}^{(2)} = 8$$

$$H_{22}^{(2)} = 0$$

Therefore, $H_{21}^{(2)}$ generates an element of the set $X_3^{(2)}$ and

$$X_3^{(2)} = \left[X_{31}^{(2)} = (8, 0, 1, 0, 0)\right]$$

$$U_3^{(2)} = 8$$

Now

$$X_3^{(2)} \cap S_1 = \emptyset \text{, therefore}$$

$$x_1 = 8, x_2 = 0 \text{ is required.}$$

solution for N(IV.2.1) and optimal value of objective function is $\frac{4}{5}$

SECTION – 4

STRONG CUT IN EPLFPP

INTRODUCTION :

In this section a procedure for solving EPLFPP is presented which unlike previous methods, discussed in Section I, Section 2, Section 3 of this chapter, avoid the investigation of some of the extreme points of convex polyhydron $RX = t$, $X \geq 0$.

THEORETICAL DEVELOPMENT :

In order to solve (IV.1.1) start with the problem (IV.1.2) viz.

$$\text{Max} \quad L(X) = \frac{CX + \alpha}{DX + \beta}$$

subject to

$$FX = f$$

$$X \geq 0$$

where $F = \begin{bmatrix} A \\ R \end{bmatrix} \qquad f = \begin{bmatrix} b \\ t \end{bmatrix}$

Assume that :

(i) Problem (IV.1.2) is bounded

(ii) $S^* = X : FX = f,\ X \geq 0$ is non-empty and bounded

(iii) $DX + \beta > 0 \quad \forall X \in S^*$

Find $Y_1^{(2)}$, the set of optimal extreme point solutions of (IV.1.2) and $V_1^{(2)}$, the value of the objective function at the elements of $Y_1^{(2)}$. As $S^* \neq \emptyset$, $Y_1^{(2)} \neq \emptyset$. Determine $Y_1^{(2)} \cap S_1$. If $Y_1^{(2)} \cap S_1 \neq \emptyset$, then every element of $Y_1^{(2)}$ is optimal for (IV.1.1). Otherwise, find $Y_2^{(2)}$, the set of second best extreme point solutions to (IV.1.2). If $Y_2^{(2)} = \emptyset$ then (IV.1.1) has no solution. If $Y_2^{(2)} \neq \emptyset$, then find $Y_2^{(2)} \cap S_1$. If $Y_2^{(2)} \cap S_1 \neq \emptyset$, then every element of $Y_2^{(2)} \cap S_1$ is optimal for (IV.1.1). If $Y_2^{(2)} \cap S_1 = \emptyset$, pick up the problem (IV.2.2) viz.

$$\text{Max} \quad L(X) \equiv \frac{CX + \alpha}{DX + \beta}$$

subject to

$$RX = t$$

$$X \geq 0$$

let $V_2^{(2)}$ be the value of the objective function of (IV.1.2) for the elements $Y_2^{(2)}$.

Now find the set $X_1^{(2)}$ of optimal extreme point solutions of (IV.2.2). As S non-empty, therefore, $X_1^{(2)} \neq \emptyset$ and as feasible region of (IV.2.2) contains feasible region of (IV.1.2), therefore, $X_1^{(2)} \cap S_1 = \emptyset$.

Find the set R_1 ($\geqslant V_2^{(2)}$) of values of objective function at the extreme points adjacent to the elements of $X_1^{(2)}$. Out of these values pick up the value, say W_1, nearest to $V_2^{(2)}$ i.e. $W_1 = \text{Min}\ (R_1)$, $W_1 \geqslant V_2^{(2)}$. Determine the extreme points which has W_1 as the value of the objective function. Find the values of objective function ($\geqslant V_2^{(2)}$) at the extreme points adjacent to the extreme points corresponding to W_1. Out of these pick up the value, say W_2, which is nearest to $V_2^{(2)}$ and find the corresponding extreme points $Rx=t$, $x \geqslant 0$ which are adjacent to extreme points corresponding to W_2 and values of the objective function at these extreme points. Out of these pick up the value, say W_3, which is nearest to $V_2^{(2)}$. This process is continued till a set of extreme points is obtained where values of the objective function are W_k ($\geqslant V_2^{(2)}$) and at all the adjacent extreme points to these extreme points, the value of the objective function is less than $V_2^{(2)}$. At this stage introduce the cut

$$L(X) \leqslant W_k$$

in problem (IV.2.2). This cut is called 'strong cut'. The problem so obtained is

$$\left.\begin{array}{l} \text{Max} \quad L(X) = \dfrac{CX + \alpha}{DX + \beta} \\ \text{subject to} \\ \qquad RX = t \\ \qquad L(X) \leq M_k \\ \qquad X \geq 0 \end{array}\right\} \qquad \text{(IV.4.1)}$$

Now extreme points solutions of (IV.3.1) are determined in a systematic order by cutting plane procedure till optimal solution of (IV.1.1) is reached or an indication of no solution of (IV.1.1) is obtained.

It may be noted that after introducing the strong cut in (IV.2.2) the extreme point sulutions of (IV.3.1) may also be ranked by enumeration technique till optimal solution of (IV.1.1) is reached or an indication of no solution is obtained.

EXAMPLE :

$$\left.\begin{array}{l} \text{Max} \quad L(X) = \dfrac{x_1 + 14x_2}{x_1 + 6x_2 + 4} \\ \text{subject to} \\ \qquad -x_1 + 2x_2 \leq 3 \\ \qquad 5x_1 + 12x_2 \leq 30 \\ \qquad x_1, x_2 \geq 0 \end{array}\right\} \qquad \text{E(IV.4.1)}$$

and (x_1, x_2) is extreme point of

$$\begin{aligned}
-2x_1 + x_2 &\leq 1 \\
-2x_1 + 2x_2 &\leq 3 \\
-2x_1 + 3x_2 &\leq 6 \\
x_2 &\leq 4 \\
9x_1 + 7x_2 &\leq 63 \\
3x_1 - x_2 &\leq 9 \\
3x_1 - 2x_2 &\leq 6 \\
x_1 - 2x_2 &\leq 1 \\
x_1, x_2 &\geqslant 0
\end{aligned}$$

SOLUTION :

In order to solve N(IV.4.1) start with the problem

$$\text{Max} \quad L(X) = \frac{x_1 + 14x_2}{x_1 + 6x_2 + 4}$$

subject to

$$FX = f \quad X \geqslant 0 \qquad \begin{aligned}
-x_1 + 2x_2 + x_3 &= 3 \\
5x_1 + 12x_2 + x_4 &= 30 \\
-2x_1 + x_2 + x_5 &= 1 \\
-2x_1 + 2x_2 + x_6 &= 3 \\
-2x_1 + 3x_2 + x_7 &= 6 \\
x_2 + x_8 &= 4
\end{aligned} \qquad N(IV.4.2)$$

$$\left[\begin{array}{l} 9x_1 + 7x_2 + x_9 = 63 \\ 3x_1 - x_2 + x_{10} = 9 \\ 3x_1 - 2x_2 + x_{11} = 6 \\ x_1 - 2x_2 + x_{12} = 1 \\ x_1, x_2, \ldots\ldots x_{12} \geq 0 \end{array}\right]$$

The set of optimal extreme point solutions of N(IV.4.2) is

$$Y_1^{(2)} = \left[Y_{11}^{(2)} = \left(\frac{12}{11}, \frac{45}{22}, 0, 0, \frac{25}{22}, \frac{12}{11}, \frac{45}{22}, \frac{43}{22}, \frac{855}{22}, \frac{171}{22}, \frac{75}{11}, 4 \right) \right]$$

$$V_1^{(2)} = \frac{327}{191}$$

$$|J = \left[r_1, r_2, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12} \right] \quad r$$

$$p = 8$$

$$|J(Y_{11}^{(2)})| = 10 > 8$$

$$\therefore Y_1^{(2)} \cap S_1 = \emptyset$$

$Y_2^{(2)}$, the set of second best extreme point solution of N(IV.4.2) is

$$Y_2^{(2)} = \left[Y_{21}^{(2)} = \left(\frac{1}{3}, \frac{5}{3}, 0, \frac{25}{3}, 0, \frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{145}{3}, \frac{29}{3}, \frac{25}{3}, 4 \right) \right]$$

$V_2^{(2)} = \frac{71}{43} = 1.65$

$|J(Y_{21}^{(2)})| = 9 > 8$

$\therefore Y_2^{(2)} \cap S_1 = \emptyset$

Hence pick up the problem

$$\left.\begin{array}{ll} & \text{Max } L(X) \geq \dfrac{x_1+14x_2}{x_1+6x_2+4} \\ & \text{subject to} \\ RX = t & \left|\begin{array}{l} -2x_1 + x_2 + x_5 = 1 \\ -2x_1 + 3x_2 + x_6 = 3 \\ -2x_1 + 3x_2 + x_7 = 6 \\ x_2 + x_8 = 4 \\ 9x_1 + 7x_2 + x_9 = 63 \\ 3x_1 - x_2 + x_{10} = 9 \\ 3x_1 - 2x_2 + x_{11} = 6 \\ x_1 - 2x_2 + x_{12} = 1 \\ x_1, x_2, x_5, x_6, \ldots x_{12} \geq 0 \end{array}\right. \\ X \geq 0 & \end{array}\right] N(IV.4.3)$$

The optional solution for N(IV.4.3) is

$X_1^{(3)} = [X_{11}^{(3)} = (3,4,0,0,3,1,0,0,8,4,5,6)]$

and

$U_1^{(3)} = \frac{59}{31}$

The value of the objective function at the extreme point adjacent to $X_{11}^{(3)}$ are

$$R_1 = \frac{59 + \frac{8x2}{9x2}}{31 + \frac{8x2}{9x2}} = \frac{539}{287} = 1.87$$

$$R_2 = \frac{59 - \frac{31}{2}}{31 - \frac{15}{2}} = \frac{87}{47} = 1.85$$

$$W_1 = \text{Min}\left[1.87, 1.85\right] = 1.85 \; (> V_2^{(2)}$$

The extreme point corresponding to W_1 is

$$\left[\left(\frac{3}{2}, 3,0,0,1,0,0,1, \frac{57}{2}, \frac{15}{2}, \frac{15}{2}, \frac{11}{2} \right) \right]$$

The value of the objective function at the extreme point adjacent to this extreme point is

$$W_2 = 1.72 \; (> V_2^{(2)}$$

The extreme point corresponding to W_2 is

$$\left[\left(\frac{1}{2}, 2, 0,0,0,0,1,2, \frac{89}{2}, \frac{19}{2}, \frac{17}{2}, \frac{9}{2} \right) \right]$$

The value of the objective function at the extreme point adjacent to this extreme point is

$$W_3 = 1.4 \; (\leq V_2^{(2)})$$

Introduce tho strong cut

$$L(X) \leqslant \frac{57}{33}$$

$$-2x_1 + 10x_2 \leqslant 19$$

in (IV.4.3). The problem so obtained is

$$\text{Max} \quad L(X) \equiv \frac{x_1 + 14x_2}{x_1 + 6x_2 + 4}$$

subject to

$$
\left.
\begin{aligned}
-2x_1 + x_2 + x_5 &= 1 \\
-2x_1 + 2x_2 + x_6 &= 6 \\
-2x_1 + 3x_2 + x_7 &= 6 \\
x_2 + x8 &= 4 \\
9x_1 + 7x_2 + x_9 &= 63 \\
3x_1 - x_2 + x_{10} &= 9 \\
3x_1 - 2x_2 + x_{11} &= 6 \\
x_1 - 2x_2 + x_{12} &= 1 \\
-2x_1 + 10x_2 + x_{13} &= 9 \\
x_1, x_2, x_5, x_6, \ldots x_{13} &\geqslant 0
\end{aligned}
\right| \qquad N(IV.4.4)
$$

The set of optimal extreme point solutions to (IV.4.4) is

$$X_1^{(4)} = \Big[X_{11}^{(4)} = \left(\tfrac{1}{2}, 2, 0, 0, 0, 0, 1, 2, \tfrac{89}{2}, \tfrac{19}{2}, \tfrac{17}{2}, \tfrac{9}{2}, 0 \right)$$

$$X_{12}^{(4)} = \left(\tfrac{49}{13}, \tfrac{23}{13}, 0, 0, \tfrac{133}{26}, 0, \tfrac{145}{26}, \tfrac{35}{26}, \tfrac{21}{2}, \tfrac{9}{26}, 0, \tfrac{33}{13}, 0 \right) \Big]$$

Now $X_1^{(4)} \cap S_1 = \emptyset$, the set of second best extreme point solution of N(IV.4.4) is

$$X_2^{(4)} = \Big[X_{21}^{(4)} = (0,1,0,0,0,1,3,3,56,10,8,3,9) \Big]$$

$$U_2^{(4)} = \frac{7}{5}$$

Now $X_2^{(4)} \cap S_1 \neq \emptyset$, therefore,

$x_1 = 0$, $x_2 = 1$, is optimal solution for N(IV.1.1) and value of objective function is $L(X) = \frac{7}{5}$

<u>EXAMPLE</u> :

Strong Cut Enumerative Procedure for solving N(IV.1.1)

<u>SOLUTION</u> :

Now $X_2^{(2)} \cap S_1 = \emptyset$

and $X_1^{(4)} \cap S_1 = \emptyset$

We have to find the set $X_2^{(4)}$ of second best extreme point solutions of N(IV.1.10)

$B_1^{(4)} = \{ B_{11}^{(4)} = (r_2, r_1, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_5)$

$B_{12}^{(4)} = (r_2, r_1, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_5) \}$

$E_1^{(4)} = \{ E_{11}^{(4)} = (r_2, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}\ r_5)$

$E_{12}^{(4)} = (r_2, r_1, r_7, r_8, r_9, r_{10}, r_6, r_{13}, r_5) \}$

Value of objective function for

$E_{11}^{(4)} = 1.4$

$E_{12}^{(4)} = 1.18$

Therefore, $E_{11}^{(4)}$ generatost an element of $X_2^{(4)}$ and

$X_2^{(4)} = \{ X_{21}^{(4)} = (0,1,0,0,0,1,3,3,56,10,8,3,9) \}$

$U_2^{(4)} = \frac{7}{5} = 1.4$

Now $X_2^{(4)} \cap S_1 \neq \emptyset$, therefore,

$x_1 = 0,\ x_2 = 1$ optimal solution and

$L(X) = \frac{7}{5}$ is the optimal value of objective function.

****

APPENDIX -

For solving EPLPP

$$\text{Max} \quad Z = CX$$

subject to

$$AX = b$$
$$X \geq 0$$

and X is extreme point of

$$DX = d$$
$$X \geq 0$$

a method is presented in chapter II where extreme points of the convex polyhydron $DX=d$, $X \geq 0$ are investigated in a systematic order till feasibility in $AX = b$ is achieved. As the constraints $AX = b$ do not come into picture in the computational aspects of the method and are only to be verified by various extreme points of the convex polyhydron $DX = d$, $X \geq 0$ it follows that whatever be the character of these constraints, say non-linear, the technique will still work. Consider the problem

$$\text{Max} \quad Z = CX$$

subject to

$$G_i(x) \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} g_i, \quad i=1,2, \ldots .n$$
$$X \geq 0$$

and X is an extreme point of

$$DX = d$$

$$X \geq 0$$

where $G_i(X)$, =, g_i, $i = 1, 2, \ldots n$ are nonlinear constraints. This problem can be solved by ranking the extreme points of convex polyhydron $DX = d$, $X \geq 0$ till an extreme point is reached which satisfies the feasibility in $G_i(X) \{\leq, =, \geq\} g_i$, $i = 1, 2, \ldots n$.

In an exactly similar manner a technique given in chapter IV can be used to solve the problem

$$\text{Max} \quad \frac{CX + \alpha}{DX + \beta}$$

subject to

$$G_i(X) \{\leq, =, \geq\} g_i, \quad i = 1, 2, \ldots n$$

X is an extreme point of

$$DX = d$$

$$X \geq 0$$

R_E_F_E_R_E_N_C_E_S

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