ON THE ASYMPTOTIC BEHAVIOUR OF SOME THREE SPECIES ECOSYSTEM MODELS

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ABSTRACT

Analysis of the asymptotic behaviour of component populations in a few three species ecosystems, viz. the one prey two predator system under the condition of no self interaction and competition for the predator populations and the two prey one predator system without self interaction and competition terms for the prey populations is done. This has been carried out by exploiting the constraint that exists in the subspace of the two populations and by using Laurent series expansions in the asymptotic region in an appropriately chosen variable. We are able to obtain the results on the asymptotic behaviour of the component populations as time tends to infinity. These behaviours are also verified by numerical analysis on the computer using the standard Runge-Kutta approximation method.

CONTENTS

CHAPTER I	INTRODUCTION	1
CHAPTER II	REVIEW OF SOME ECOSYSTEM MODELS	4
CHAPTER III	RESULTS ON SOME THREE SPECIES LOTKA-VOLTERRA MODELS IN THE ASYMPTOTIC REGION	19
	SECTION I - ONE PREY-TWO PREDATOR SYSTEM	19a
	SECTION II - TWO PREY-ONE PREDATOR SYSTEM	35
CHAPTER IV	ILLUSTRATION OF THE ANALYTICAL RESULTS USING RUNGE-KUTTA APPROXIMATION METHOD	46
CHAPTER V	SUMMARY OF THE RESULTS	56
FIGURES		59
APPENDIX I:	PROGRAME LISTING OF THE ONE PREY-TWO PREDATOR SYSTEM	65
APPENDIX II:	PROGRAME LISTING OF THE TWO PREY-ONE PREDATOR SYSTEM	68
REFERENCES		71

CHAPTER - I

INTRODUCTION

The study of the three species ecosystem models occupies an important place in theoretical ecology. The elucidation of these models will lead to clues to an understanding of the more complex multispecies systems. The ecosystem models, as described by a set of differential equations, are in general non-linear. Due to the nonlinearity, it is very difficult to judge the exact behaviour of the component populations in the long run, as usually the non-linear equations can not be solved exactly.

A great deal of work has been done on three species models. The works of Parrish and Saila (1970), Cramer and May (1972) and Bhat and Pande (1980, 1981) are notable in this context. The implications of the result of a three step prey-predator food chain (Bhat and Pande, 1981) are quite interesting. In the model three populations N_1, N_2 and N_3 are considered, with N_2 preying on N_1 and N_3 preying on N_2. The model contained the prey-predator interactions and self interaction for the population N_2. All the interactions were taken to be of the Lotka-Volterra form. Due to nonlinearity of the equations, the model was not solvable analytically. However, the behaviour of the component populations was described using numerical methods for a certain range of parameters occurring in the model. It was found that both N_1 and N_3 rose indefinitely while N_2 reached a finite constant value asymptotically. Even though the results are quite satisfactory, the lack of an analytical base is felt.

Varma and Pande (1986) first tried to give some strong analytical base to the above results. Although they were not able to get the exact solutions, they obtained analytically the behaviour of the populations in the asymptotic region as $t \longrightarrow \infty$.

In the present dissertation, we extend the work of the above authors to the one prey-two predator system and the two prey-one predator system. In the case of one prey-two predator system the self interaction and competition terms are excluded for the predator populations, whereas in case of the two prey-one predator system the self-interaction and compeitition terms for the prey populations are excluded. These results give the earlier results a more strong analytical base.

Our results have been obtained by exploring a constraint that exists in the subspace of two populations, and by using suitable Laurent series expansions in the asymptotic region for an appropriately chosen variable. We are able to obtain results on the asymptotic behaviour of the three species. The precise conditions pertaining to the asymptotic behaviour are also obtained. The method used for the purpose is quite simple and has got reasonably good applicability.

All the results obtained in the above manner are verified by numerical analysis on the computer. The verification has been carried out on H.P. 9836A computer, using the Runge-Kutta approximation method.

CHAPTER - II

REVIEW OF SOME ECOSYSTEM MODELS

We shall build up the three species ecosystem model step by step, starting with the single species system, and analyse it explicitely in this chapter. The latter is the simplest of possible systems realised only under extremely special conditions. Let us assume an "unlimited environment". It can be further assumed that the individuals have no effect on one another, and that the rate of growth per individual is the same for all individuals and is a constant in time. If we denote this rate by α and the population by $N_1(t)$, then the dynamics of this system is given by the equation :

$$\frac{dN_1}{dt} = \alpha N_1 \tag{1.1}$$

which has the simple solution,

$$N_1(t) = N_1(o)\, e^{\alpha t} \tag{1.2}$$

where $N_1(o)$ is the population at time $t = 0$.

This is the well known Malthusian picture of population growth where the population rises exponentially with time (Pielou, 1977).

But in reality the environment is not an unlimited one. The food available to the population is sooner or later going to get limited because of the rising population. Pearl-Verhulst suggested a modification of α to $(\alpha - \beta N_1)$ which leads to a fall in the rate with increase in population. The equation, then, is :

$$\frac{dN_1}{dt} = (\alpha - \beta N_1)N_1 \tag{1.3}$$

and the solution to this "Pearl-Verhulst logistic equation" is:

$$N_1(t) = \frac{\alpha/\beta}{1 + e^{-\alpha(t-t_o)}} \tag{1.4}$$

where the constant $e^{\alpha t_o}$ is given in terms of the initial population $N_1(o)$ by,

$$e^{\alpha t_o} = \frac{(\alpha/\beta) - N_1(o)}{N_1(o)} \tag{1.5}$$

The solution has an asymptotic value as $t \longrightarrow \infty$, which is α/β. The value $N_1 = \alpha/\beta$ is the maximum that the population can reach and is therefore called the "carrying capacity" of the given environment.

Now we consider that there are two populations N_1 and N_2 such that N_1 take its food directly from the environment, as in the earlier models, but N_2 derive its food from N_1 only. The presence of N_2 thus affects the growth rate α. Considering the simplest possibility we replace α by $(\alpha - \lambda_1 N_2)$, where λ_1 is a positive constant. So we get,

$$\frac{dN_1}{dt} = (\alpha - \lambda_1 N_2)\, N_1 \tag{1.6}$$

The second term on the right hand side in this equation describes the interaction between the two populations. Such an interaction term should clearly also govern the rate of change of the population N_2, but the contribution should now be positive. We thus have,

$$\frac{dN_2}{dt} \propto \lambda_2 N_1 N_2 \tag{1.7}$$

where λ_2 is a positive constant. If the population N_2 is left to itself, it should obviously die out. Assuming that the decay rate per individual, say γ; is a constant in time and is the same for all individuals, we immediately have,

$$\frac{dN_2}{dt} \propto -\gamma N_2 \tag{1.8}$$

where γ is again positive. The complete equation for the evolution of the population N_2 can therefore be written as:

$$\frac{dN_2}{dt} = -\gamma N_2 + \lambda_2 N_1 N_2 \tag{1.9}$$

This system, given by equation (1.6) and (1.9) is the well known Lotka-Volterra model (Pielou, 1977), describing a two species prey-predator system.

Equations (1.6) and (1.9) are coupled nonlinear equations which cannot be solved analytically. We have to consider some approximations and with the help of numerical methods we can solve them. In view of its nonlinear nature, it is unlikely that the full information content of this system is uncovered by such methods. [It may be noted that equation (1.3) is also nonlinear. However, its simple form enables us to solve it exactly by direct integration]. However, an exact result, can be established. This was done originally by Volterra (1927). Volterra observed that the system possesses a conserved quantity, using which it can be proved that the system traces closed trajectories in the $N_1 - N_2$ phase space. This shows that N_1 and N_2 are oscillatory as functions of t, implying their continued co-existence.

Arguments similar to those used in constructing the Lotka-Volterra model can also be used for two species systems where the two species are no more prey and predator, but instead, both derive their food directly from the environment and compete with each other for the same. We simply keep positive signs for the first terms on the right hand sides in equations (1.6) and (1.9) and keep negative signs for both the interaction terms. It is possible that the growth of the two populations can also be influenced by "self-interaction" as in the case of equation (1.3). Incorporating that also, we have,

$$\frac{dN_1}{dt} = \varepsilon_1 N_1 - \alpha_1 N_1^2 - \beta_1 N_1 N_2$$

$$\frac{dN_2}{dt} = \varepsilon_2 N_2 - \alpha_2 N_1 N_2 - \beta_2 N_2^2 \qquad (1.10)$$

where all the parameters ε_1, α_1, β_1 and ε_2, α_2, β_2 are positive constants.

This is the well known Gause-Witt model for the two competing species. Here also the nonlinear nature of these coupled equations make it difficult to solve them exactly. It is possible, however, to show that this system does possess stable equilibrium under certain conditions given by certain inequality relations between the various parameters involved. This may be achieved by graphical methods using isoclines. Another approach is to consider the linearised version of the equations in the neighbourhood of the equilibrium points and to use the so called Hurwitz-Routh criteria.

It is straight forward to generalize the above ideas to incorporate more than two species either with prey-predator interactions or with competition. One can also construct models wherein some pairs have prey-predation relationships and the others have only competition. It is quite simple, then, to write the full structure of the general K-species model.

But as reported earlier, the main difficulty in this approach is to solve these coupled nonlinear equations without any approximation. The numerical analysis that we may perform for different points or even regions of the parameter space, will never give us the full information content of these equations. It is thus important to construct models which are more tractable, hopefully even exactly solvable.

Let us consider the form $(\alpha - \beta \log N_1)$ (Gompertz, 1825; Gomantam, 1974). Equation (1.3) is then replaced by,

$$\frac{dN_1}{dt} = (\alpha - \beta \log N_1)\, N_1 \tag{1.11}$$

which has the solution,

$$N_1(t) = e^{\alpha/\beta} \exp\,[\{\log N_1(o) - \alpha/\beta\}\, e^{-\beta t}] \tag{1.12}$$

The solution is capable of yielding the same kind of population growth as we find in the Pearl-Verhulst model, the expression for the carrying capacity now being $e^{\alpha/\beta}$.

In a similar way, the inhibition of the growth rate α for the population N_1 due to its interaction with population N_2, may also be considered in the form ($\alpha - \lambda_1 \log N_2$) instead of ($\alpha - \lambda N_2$). The growth rate for N_2 can also be modified to ($-\beta + \lambda_2 \log N_1$) in place of ($-\beta + \lambda_2 N_1$). We thus get the following coupled equations to describe an interacting two species prey-predator system.

$$\frac{dN_1}{dt} = \alpha N_1 - \lambda_1 N_1 \log N_2$$

$$\frac{dN_2}{dt} = -\beta N_2 + \lambda_2 N_2 \log N_1 \qquad (1.13)$$

This system of nonlinear equations can be solved exactly.

This model with "logarithemic" interaction terms which we may call the Gompertz model can easily be generalised to cover the Gause-Witt case and the results are quite satisfactory. It is interesting to note that this approach can cover various multi species interacting systems, with its solvability remaining intact.

Now we discuss the Gompertz model for some of the three species ecosystems. For instance we consider the one prey-two predator system (Bhat and Pande, 1983). Let the prey population be denoted by N_1 and the predator populations by N_2 and N_3. The time development of these populations will be governed :

(i) by natural growth (for N_1) and decay (for N_2 and N_3) terms, which in the absence of any interactions will lead to the usual exponential rise for the prey and exponential fall for the predators, and

(ii) by the various self interaction and mutual interaction terms. All these interaction terms are written in the Gompertz form. The equations describing the model are,

$$\dot{N}_1 = \epsilon_1 N_1 - \alpha_1 N_1 \log N_1 - \beta_1 N_1 \log N_2 - \gamma_1 N_1 \log N_3$$

$$\dot{N}_2 = -\epsilon_2 N_2 + \alpha_2 N_2 \log N_1 - \beta_2 N_2 \log N_2 - \gamma_2 N_2 \log N_3 \qquad (1.14)$$

$$\dot{N}_3 = -\epsilon_3 N_3 + \alpha_3 N_3 \log N_1 - \beta_3 N_3 \log N_2 - \gamma_3 N_3 \log N_3$$

where $\dot{N}_1$, $\dot{N}_2$ and $\dot{N}_3$ stand for the respective time derivatives. The signs of various terms depend on whether they represent self interaction, competition or prey-pedation. The sign is negative for the former two, and as for the latter, the term has a negative sign in the equation for the time development of the prey population and positive sign in the corresponding equation for the predator population. The ϵ_i terms are here the natural growth and decay terms; those carrying the constants α_1, β_1 and γ_3 are self interaction terms; and γ_2 and β_3 terms represent competition between the two predator populations and the remaining terms represent the prey-predator interactions.

Introducing the notation,

$X_1 = \log N_1; \quad X_2 = \log N_2; \quad X_3 = \log N_3$

we can rewrite equations (1.14) as,

$$
\begin{aligned}
\dot{X}_1 &= \varepsilon_1 - \alpha_1 X_1 - \beta_1 X_2 - \gamma_1 X_3 \\
\dot{X}_2 &= -\varepsilon_2 + \alpha_2 X_2 - \beta_2 X_2 - \gamma_2 X_3 \\
\dot{X}_3 &= -\varepsilon_3 + \alpha_3 X_1 - \beta_3 X_2 - \gamma_3 X_3
\end{aligned}
\tag{1.15}
$$

The above model yields solutions which can possess stable equilibrium, implying co-existence of all the three species.

The above was the general situation where we considered all the different types of interactions. It is of much interest to see what happens when some of the above interactions are absent. We take for instance the case with no competition and self interaction for the predators. So we have

$$\beta_2 = \gamma_3 = \gamma_2 = \beta_3 = 0$$

Thus equations (1.15) reduce to,

$$
\begin{aligned}
\dot{X}_1 &= \varepsilon_1 - \alpha_1 X_1 - \beta_1 X_2 - \gamma_1 X_3 \\
\dot{X}_2 &= -\varepsilon_2 + \alpha_2 X_1 \\
\dot{X}_3 &= -\varepsilon_3 + \alpha_3 X_1
\end{aligned}
\tag{1.16}
$$

We solve these equations by differentiating once the first equation and substituting from second and third the values of $\dot{X}_2$ and $\dot{X}_3$. So we get,

$$\ddot{X}_1 = A - BX_1 - \alpha_1 \dot{X}_1 \qquad (1.17)$$

where,

$A = \beta_1 \varepsilon_2 + \gamma_1 \varepsilon_3$ and

$B = \beta_1 \alpha_2 + \gamma_1 \alpha_3$

Equation (1.17) is a nonhomogeneous linear equation, the full solution of which is,

$$X_1 = \frac{A}{B} + D_1 e^{E_1 t} + D_2 e^{E_2 t} \qquad (1.18)$$

where D_1 and D_2 are two arbitrary constants and,

$$E_1 = \frac{[-\alpha_1 + (\alpha_1^2 - 4B)^{1/2}]}{2}$$

$$E_2 = \frac{[-\alpha_1 - (\alpha_1^2 - 4B)^{1/2}]}{2} \qquad (1.19)$$

when E_1 and E_2 are complex, we have $E_1^* = E_2$ and $D_1^* = D_2$. For real E_1 and E_2; D_1 and D_2 are also real.

Substituting the values of X_1 from equation (1.18) in the last two equations of (1.16) and integrating we obtain,

$$X_2 = C_1 + Kt + \alpha_2 \left[\frac{D_1}{E_1} e^{E_1 t} + \frac{D_2}{E_2} e^{E_2 t} \right] \quad (1.20)$$

$$X_3 = C_2 - \frac{\beta_1}{\gamma_1} Kt + \alpha_3 \left[\frac{D_1}{E_1} e^{E_1 t} + \frac{D_2}{E_2} e^{E_2 t} \right] \quad (1.21)$$

where,

$$K = \frac{\gamma_1 [\alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2]}{B} \quad \text{and}$$

C_1 and C_2 are two integration constants connected by,

$$\beta_1 C_1 - \gamma_1 C_2 + \alpha_1 \frac{A}{B} - \varepsilon_1 = 0, \quad (1.22)$$

which is obtained when the expressions for X_1, X_2 and X_3 are substituted in the first equation in (1.16).

It is clear from equation (1.19) that E_1 and E_2 always have negative real parts. Therefore, X_1 (and hence N_1) is always finite and non-vanishing. For $t \to \infty$, it acquires the value,

$$X_1 (t \to \infty) = \frac{A}{B} \quad (1.23)$$

As regards X_2 and X_3, due to the presence of the term linear in t, as $t \to \infty$, one of the predator populations blow up and the other vanishes. Clearly, under the condition

$$(\alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2) > 0, \qquad (1.24)$$

$N_2(t \longrightarrow \infty) \longrightarrow \infty$

$N_3(t \longrightarrow \infty) \longrightarrow 0$

and under the condition

$$(\alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2) < 0, \qquad (1.25)$$

$N_2(t \longrightarrow \infty) \longrightarrow 0$

$N_3(t \longrightarrow \infty) \longrightarrow \infty$

For both N_2 and N_3 to remain finite and coexist, the constraint

$$K = 0 \Rightarrow (\alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2) = 0, \qquad (1.26)$$

or simply $\alpha_2/\varepsilon_2 = \alpha_3/\varepsilon_3$, has to be satisfied.

In that case,

$$X_2 (t \longrightarrow \infty) = C_1$$
$$X_3 (t \longrightarrow \infty) = C_2 \qquad (1.27)$$

we get a very similar result in the case of two prey-one predator system when we exclude competition and self interaction for the prey populations. As $t \longrightarrow \infty$, one of the prey populations blow up and the other vanishes, whereas under the constraint $K = 0$ all the three populations coexist.

The above system can also be discussed within the Lotka-Volterra model, with the prey population denoted by N_1 and the predator populations by N_2 and N_3. The dynamics of the system for the case with no competition and self interaction for predators is then given by,

$$\dot{N}_1 = \epsilon_1 N_1 - \alpha_1 N_1^2 - \beta_1 N_1 N_2 - \gamma_1 N_1 N_3$$

$$\dot{N}_2 = -\epsilon_2 N_2 + \alpha_2 N_2 N_1 \qquad (1.28)$$

$$\dot{N}_3 = -\epsilon_3 N_3 + \alpha_3 N_3 N_1$$

Assuming all $N_i > 0$, we obtain the equilibrium value $\bar{N}_i$ from:

$$\epsilon_1 - \alpha_1 \bar{N}_1 - \beta_1 \bar{N}_2 - \gamma_1 \bar{N}_3 = 0 \qquad (1.29)$$

$$-\epsilon_1 + \alpha_2 \bar{N}_1 = 0$$
$$-\epsilon_3 + \alpha_3 \bar{N}_1 = 0 \qquad (1.30)$$

Equation (1.30) gives,

$$\bar{N}_1 = \frac{\epsilon_2}{\alpha_2} = \frac{\epsilon_3}{\alpha_3} \qquad (1.31)$$

The possibility of all populations remaining finite and non-vanishing cannot be ruled out. But in view of the lack of exact solution for equation (1.28) nothing definite can be said analytically in this regard. But when we look at the results obtained by numerical analysis under the conditions :

(i) $\frac{\varepsilon_2}{\alpha_2} = \frac{\varepsilon_3}{\alpha_3}$

(ii) $\frac{\varepsilon_2}{\alpha_2} > \frac{\varepsilon_3}{\alpha_3}$ and

(iii) $\frac{\varepsilon_2}{\alpha_2} < \frac{\varepsilon_3}{\alpha_3}$

We see the following :-

Under condition (i) there is co-existence of all the three populations. Under condition (ii), the population N_2 steadily vanishes while population N_1 and N_3 oscillate with decreasing amplitude about a finite value at which they finally settle. Under condition (iii) N_3 vanishes and N_1 and N_2 reach certain finite values.

Thus, we see that the results in the Lotka-Volterra model are very similar to what we obtained in the Gompertz model. They are identical as to which populations survive and which one dies out, but in place of the indefinite rise of one of the surviving populations in the Gompertz model, we now have the corresponding population reaching a finite constant value. That is the situation as regards case (ii) and (iii). The results in case (i) are totally similar in the two cases. Similar agreement between the results of the Lotka-Volterra model and those of the

Gompertz model is also obtained for the two prey-one predator case (Ph.D thesis: Bhat, 1980). In fact, the main purpose in discussing in detail the solvable Gompertz model was to obtain some guidelines as to what kind of numerical solutions to expect in the Lotka-Volterra case under different conditions. The problem of obtaining more general results analytically in case of the Lotka-Volterra model, of course, remains unsolved.

In this dissertation we are able to obtain the behaviour of the three species systems analytically in the asymptotic region as t $\longrightarrow \infty$. We again deal with the cases when competetion and self interaction for the predators is excluded in the one prey-two predator system and when the competition and self interaction for the prey is excluded in the two prey-one predator system. The details of our approach and our results are presented in the next chapter. In the chapter following that we present some numerical examples done in the computer, which illustrate the analytically obtained results of the earlier chapter.

The approach followed in obtaining the analytical results of the next chapter was first used by Varma and Pande (1986).

CHAPTER - III

RESULTS ON SOME THREE SPECIES LOTKA-VOLTERRA MODELS IN THE ASYMPTOTIC REGION

In this chapter we carry out an analysis of certain three species ecosystems within the Lotka-Volterra model. In Section I below we consider the one prey-two predator system in which competition and self interaction terms are excluded for the predator populations. In Section II we deal with the two prey-one predator system and in this case we do not consider self interaction and competition terms for the prey populations.

It is not possible to write the exact solutions of the above systems. However, important information about the populations can be ascertained by analysing the behaviour of the systems in the asymptotic region as $t \longrightarrow \infty$. The results are obtained by exploring the constraint that exists in the subspace of the two populations and using suitable Laurent series expansions in an appropriately chosen variable in the asymptotic region. We also illustrate, in the next chapter, our analytical results with numerical calculations done in the Computer.

SECTION - I

ONE PREY-TWO PREDATOR SYSTEM

We now consider the one prey-two predator system. Let the prey population be denoted by N_1 and the predator populations by N_2 and N_3. The system under consideration is described by the following set of equations :

$$\dot{N}_1 = \epsilon_1 N_1 - \alpha_1 N_1^{\,2} - \beta_1 N_1 N_2 - \gamma_1 N_1 N_3$$

$$\dot{N}_2 = -\epsilon_2 N_2 + \alpha_2 N_2 N_1 \qquad (2.1)$$

$$\dot{N}_3 = -\epsilon_3 N_3 + \alpha_3 N_3 N_1$$

where all the parameters ϵ_1, α_1, β_1, γ_1, ϵ_2, α_2, ϵ_3 and α_3 are positive and the dots on the N's signify time derivatives. Let us define a variable Z such that $Z = e^{\delta t}$, where $\delta > 0$. The above equations in terms of Z can be written as :

$$\delta Z \frac{dN_1}{dZ} = \epsilon_1 N_1 - \alpha_1 N_1^{\,2} - \beta_1 N_1 N_2 - \gamma_1 N_1 N_3$$

$$\delta Z \frac{dN_2}{dZ} = -\epsilon_2 N_2 + \alpha_2 N_1 N_2 \qquad (2.2)$$

$$\delta Z \frac{dN_3}{dZ} = -\epsilon_3 N_3 + \alpha_3 N_1 N_3$$

From second equation of (2.2), we get,

$$\delta Z \frac{1}{N_2} \frac{dN_2}{dZ} = -\varepsilon_2 + \alpha_2 N_1$$

or $$N_1 = \frac{1}{\alpha_2} \left[\delta Z \frac{1}{N_2} \frac{dN_2}{dZ} + \varepsilon_2 \right] \qquad (2.3)$$

Similarly, we have from third equation of (2.2),

$$N_1 = \frac{1}{\alpha_3} \left[\delta Z \frac{1}{N_3} \frac{dN_3}{dZ} + \varepsilon_3 \right] \qquad (2.4)$$

Equating equations (2.3) and (2.4), we have,

$$\frac{1}{\alpha_2} \left[\delta Z \frac{dN_2}{dZ} \frac{1}{N_2} + \varepsilon_3 \right] = \frac{1}{\alpha_3} \left[\delta Z \frac{1}{N_3} \frac{dN_3}{dZ} + \varepsilon_3 \right]$$

or, $$\alpha_3 \frac{dN_2}{N_2} \frac{Z}{dZ} - \alpha_2 \frac{dN_3}{N_3} \frac{Z}{dZ} = \frac{\alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2}{\delta}$$

or, $$\alpha_3 \frac{dN_2}{N_2} - \alpha_2 \frac{dN_3}{N_3} = \frac{k}{\delta} \frac{dZ}{Z} \qquad (2.5)$$

where,

$$k = \alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2 \qquad (2.6)$$

Integrating equation (2.5) we have,

$$\alpha_3 \log N_2 - \alpha_2 \log N_3 = (k/\delta) \log Z + \log A$$

$$\text{or,} \quad \log \frac{N_2^{\alpha_3}}{N_3^{\alpha_2}} = \log A\, Z^{k/\delta}$$

$$\text{or,} \quad \frac{N_2^{\alpha_3}}{N_3^{\alpha_2}} = A\, Z^{k/\delta} , \qquad (2.7)$$

where, A is a constant determined by the initial conditions.

In view of the self interaction term present in the first equation of (2.2) which generally leads to frictional damping and saturation (Volterra, 1927) we look for a solution of the system of equations such that $N_1 \longrightarrow$ constant as $t \longrightarrow \infty$, or in view of the positivity of δ, we look for,

$$\lim_{Z \to \infty} N_1(Z) = a_o \qquad (2.8)$$

where a_o is a constant. This would imply around $Z = \infty$ the following Laurent expansion for $N_1(Z)$:

$$N_1(Z) = a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \qquad (2.9)$$

We then have,

$$\lim_{Z \to \infty} Z \frac{d}{dZ} (\log N_1) = 0 \qquad (2.10)$$

Now first equation of (2.2) can be written as,

$$\delta Z \frac{d}{dZ} (\log N_1) = \varepsilon_1 - \alpha_1 N_1 - \beta_1 N_2 - \gamma_1 N_3 \qquad (2.11)$$

Using equation (2.10), we get,

$$\lim_{Z \to \infty} (\varepsilon_1 - \alpha_1 N_2 - \beta_1 N_2 - \gamma_1 N_3 = 0$$

or, $$\lim_{Z \to \infty} (\beta_1 N_2 + \gamma_1 N_3) = \varepsilon_1 - \alpha_1 a_o \qquad (2.12)$$

$$\equiv C$$

where C is a constant. Thus the Laurent expansions of $N_2(Z)$ and $N_3(Z)$ around $Z = \infty$ should be,

$$N_2(Z) = b_o + \gamma_1 f(Z) + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \qquad (2.13)$$

$$N_3(Z) = c_o - \beta_1 f(Z) + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \qquad (2.14)$$

where f(Z) will be a polynomial in Z with some leading power Z^m, where $m > 0$. The above general results will satisfy equation (2.12). However, in view of the fact that our populations should always be positive, i.e., $N_2(Z)$, $N_3(Z) > 0$ for all $Z > 0$, we must have f(Z) identically equal to zero. This is because otherwise, at least for very large Z, where the leading terms will be coming from f(Z), either $N_2(Z)$ [when f(Z) is negative] or $N_3(Z)$ [when f(Z) is positive] will become negative. We thus conclude that the desired expansions for $N_2(Z)$ and $N_3(Z)$ have to be,

$$N_2(Z) = b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \tag{2.15}$$

$$N_3(Z) = c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \tag{2.16}$$

Substituting equations (2.15) and (2.16), equation (2.7) can now be written in the form,

$$\frac{\left[b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \right]^{\alpha_3}}{\left[c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \right]^{\alpha_2}} = A\, Z^{k/\delta} \tag{2.17}$$

Three cases now arise corresponding to $k > 0$, $k < 0$ and $k = 0$ We consider them one by one.

CASE - I: - When $k > 0$.

Since $k > 0$, the right hand side of equation (2.17) tends to ∞ for $Z \longrightarrow \infty$, whereas on the left hand side we are left with the ratio of numerator and denominator which is a constant. Thus, for right hand side to be infinity we should put $c_o = 0$ and then $\sum_{n=1}^{\infty} c_{-n} Z^{-n}$ will contribute for the positive powers of Z when it goes to the numerator. Thus we are left with the following expansions, for $N_2(Z)$ and $N_3(Z)$ as $Z \longrightarrow \infty$.

$$N_2(Z) = b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \tag{2.18}$$

$$N_3(Z) = \sum_{n=i}^{\infty} c_{-n} Z^{-n} \tag{2.19}$$

Substituting equations (2.9), (2.18) and (2.19) in the first equation of (2.2), we obtain

$$\varepsilon_1 [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}] - \alpha_1 [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}] [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]$$

$$- \beta_1 [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}] [b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}]$$

$$- \gamma_1 [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}] [\sum_{n=i}^{\infty} c_{-n} Z^{-n}]$$

$$= \delta Z \frac{d}{dZ} [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]$$

$$= \delta Z [\sum_{n=1}^{\infty} (-n) a_{-n} Z^{-n-1}] \tag{2.20}$$

Substituting equations (2.9), (2.18) and (2.19) in the second equation of (2.2), we have,

$$-\epsilon_2 [b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}] + \alpha_2 [b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}] [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]$$

$$= \delta Z \frac{d}{dZ} [b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}]$$

$$= \delta Z [\sum_{n=1}^{\infty} (-n) b_{-n} Z^{-n-1}] \qquad (2.21)$$

Lastly, substituting equations (2.9), (2.18) and (2.19), in the last equation of (2.2), we get,

$$-\epsilon_3 [\sum_{n=i}^{\infty} c_{-n} Z^{-n}] + \alpha_3 [\sum_{n=i}^{\infty} c_{-n} Z^{-n}] [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]$$

$$= \delta Z \frac{d}{dZ} [\sum_{n=i}^{\infty} c_{-n} Z^{-n}]$$

$$= \delta Z [\sum_{n=i}^{\infty} (-n) c_{-n} Z^{-n-1}] \qquad (2.22)$$

Equating the coefficients of like powers of Z we obtain from (2.20),

$$\epsilon_1 a_o - \alpha_1 a_o^2 - \beta_1 a_o b_o = 0 \qquad (2.23)$$

From (2.21), we have

$$-\epsilon_2 b_o + \alpha_2 b_o a_o = 0 \qquad (2.24)$$

Thus we have from (2.24)

$$a_o = \frac{\varepsilon_2}{\alpha_2} \tag{2.25}$$

And we have from (2.23)

$$\varepsilon_1 a_o - \alpha_1 a_o^2 - \beta_1 a_o b_o = 0 \tag{2.26}$$

or $\varepsilon_1 - \alpha_1 a_o - \beta_1 b_o = 0$

or $\beta_1 b_o = \varepsilon_1 - \dfrac{\alpha_1 \varepsilon_2}{\alpha_2}$

Here we put the value of a_o obtained from equation (2.25). Thus,

$$b_o = \frac{\varepsilon_1 \alpha_2 - \varepsilon_2 \alpha_1}{\beta_1 \alpha_2} \tag{2.27}$$

Equation (2.7) yields,

$$[b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}]^{\alpha_3} [\sum_{n=i}^{\infty} c_{-n} Z^{-n}]^{-\alpha_2} = A Z^{k/\delta}$$

or $b_o^{\alpha_3} (c_{-i} Z^{-i})^{-\alpha_2} = A Z^{k/\delta}$

Rest of the terms vanishes in the limit $Z \rightarrow \infty$. So we have,

$i \alpha_2 = k/\delta$

$$\text{or,} \quad \delta = k / i\alpha_2 \tag{2.28}$$

Reverting to the variable t, we thus obtain the following asymptotic behaviour for the three populations $N_1(t)$, $N_2(t)$ and $N_3(t)$.

$$\lim_{t \to \infty} N_1(t) = a_o = \varepsilon_2/\alpha_2$$

$$\lim_{t \to \infty} N_2(t) = b_o = \frac{\varepsilon_1\alpha_2 - \varepsilon_2\alpha_1}{\beta_1\alpha_2} \qquad (2.29)$$

$$\lim_{t \to \infty} N_3(t) = c_{-i}\, e^{-(k/\alpha_2)t} \longrightarrow 0$$

Thus, from the above equations we come to the conclusion that the prey population N_1 uniquely goes to the value ε_2/α_2 as $t \to \infty$, in which case one of the predator population N_2 tends to the value $\frac{(\varepsilon_1\alpha_2 - \varepsilon_2\alpha_1)}{\beta_1\alpha_2}$ and the other predator population N_3 vanishes exponentially. The constant c_{-i} is determined by the requirement that,

$$\frac{b_o^{\alpha_3}}{(c_{-i})^{\alpha_2}} = A \qquad (2.30)$$

where A is a constant appearing in equation (2.7) and is determined by the initial conditions.

CASE - II: - When $k < 0$.

As $k < 0$, the right hand side of equation (2.17) tends to zero for $Z \longrightarrow \infty$, whereas on the left hand side we are again left with the ratio of numerator and denominator which is a constant. So in this case for right hand side to be zero we should put $b_o = 0$ and then $\sum_{n=1}^{\infty} b_{-n} Z^{-n}$ will contribute for the powers of Z. Thus we have the following expansions for $N_2(Z)$ and $N_3(Z)$, as $Z \longrightarrow \infty$.

$$N_2(Z) = \sum_{n=q}^{\infty} b_{-n} Z^{-n} \tag{2.31}$$

$$N_3(Z) = c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \tag{2.32}$$

whereas the expansion for $N_1(Z)$ remains as usual as in equation (2.9).

Substituting equations (2.9), (2.31) and (2.32) in first equation of (2.2), we obtain

$$\epsilon_1 \left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}\right] - \alpha_1 \left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}\right] \left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}\right]$$

$$- \beta_1 \left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}\right] \left[\sum_{n=q}^{\infty} b_{-n} Z^{-n}\right]$$

$$- \gamma_1 \left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}\right] \left[c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n}\right]$$

$$= \delta Z \frac{d}{dZ} \left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}\right]$$

$$= \delta Z \left[\sum_{n=1}^{\infty} (-n)\, a_{-n} Z^{-n-1} \right] \qquad (2.33)$$

Again, substituting equations (2.9), (2.31) and (2.32) in the second equation of (2.2) we obtain

$$-\varepsilon_2 \left[\sum_{n=q}^{\infty} b_{-n} Z^{-n} \right] + \alpha_2 \left[\sum_{n=q}^{\infty} b_{-n} Z^{-n} \right] \left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \right]$$

$$= \delta Z \frac{d}{dZ} \left[\sum_{n=q}^{\infty} b_{-n} Z^{-n} \right]$$

$$= \delta Z \left[\sum_{n=q}^{\infty} (-n)\, b_{-n} Z^{-n-1} \right] \qquad (2.34)$$

And lastly substituting equations (2.9), (2.31) and (2.32) in the last equation of (2.2), we get

$$-\varepsilon_3 \left[c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \right] + \alpha_3 \left[c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \right] \left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \right]$$

$$= \delta Z \frac{d}{dZ} \left[c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \right]$$

$$= \delta Z \left[\sum_{n=1}^{\infty} (-n)\, c_{-n} Z^{-n-1} \right] \qquad (2.35)$$

Equating the coefficients of like powers of Z we obtain from equation (2.33)

$$\varepsilon_1 a_o - \alpha_1 a_o^2 - \gamma_1 a_o c_o = 0 \qquad (2.36)$$

From equation (2.34) we have,

$$-\epsilon_3 c_o + \alpha_3 c_o a_o = 0 \tag{2.37}$$

From this equation, we get

$$a_o = \epsilon_3 / \alpha_3 \tag{2.38}$$

And from equation (2.35) we have,

$$\epsilon_1 a_o - \alpha_1 a_o^2 - \gamma_1 a_o c_o = 0$$

$$\text{or} \quad \gamma_1 c_o = \epsilon_1 - \alpha_1 a_o$$

$$= \epsilon_1 - \frac{\alpha_1 \epsilon_3}{\alpha_3}$$

$$\text{or,} \quad c_o = \frac{\epsilon_1 \alpha_3 - \alpha_1 \epsilon_3}{\alpha_3 \gamma_1} \tag{2.39}$$

Equation (2.7) yields in the similar way as in the last case,

$$\delta = -\frac{k}{q\,\alpha_3} \tag{2.40}$$

Reverting to the variable t, we thus obtain the following asymptotic behaviour for the three populations $N_1(t)$, $N_2(t)$ and $N_3(t)$ as $t \longrightarrow \infty$,

$$\lim_{t \to \infty} N_1(t) = a_o = \frac{\epsilon_3}{\alpha_3}$$

$$\lim_{t \to \infty} N_2(t) = b_{-q}\, e^{(k/\alpha_3)t} \longrightarrow 0 \tag{2.41}$$

$$\lim_{t \to \infty} N_3(t) = c_o = \frac{\epsilon_1 \alpha_3 - \alpha_1 \epsilon_3}{\alpha_3 \gamma_1}$$

Thus, from the above equation we again find that the prey population N_1 uniquely goes to the value ϵ_3/α_3 as $t \rightarrow \infty$, while the predator population N_3 tends to the value $\frac{\epsilon_1 \alpha_3 - \alpha_1 \epsilon_3}{\alpha_3 \gamma_1}$. and N_2 vanishes exponentially. The constant b_{-q} is determined by the requirement

$$\frac{(b_{-q})^{\alpha_3}}{c_o^{\alpha_2}} = A \qquad (2.42)$$

where A is a constant determined by the initial conditions.

CASE - III :- When k = 0.

As k = 0, the right hand side of equation (2.17) reduces to A as $Z \rightarrow \infty$. So in this case we have the following expansions for $N_2(Z)$ and $N_3(Z)$,

$$N_2(Z) = b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \qquad (2.43)$$

$$N_3(Z) = c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \qquad (2.44)$$

and the expansion for $N_1(Z)$ remains the same as in equation (2.9).

Substituting equations (2.9), (2.43) and (2.44) in the first equation of (2.2), we get

$$\epsilon_1 [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}] - \alpha_1 [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}] [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]$$

$$- \beta_1 [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}] [b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}]$$

$$- \gamma_1 [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}] [c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n}]$$

$$= \delta Z \frac{d}{dZ} [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]$$

$$= \delta Z [\sum_{n=1}^{\infty} (-n) a_{-n} Z^{-n-1}] \qquad (2.45)$$

Again stustituting equations (2.9), (2.43) and (2.44) in the second equation of (2.2) we have,

$$-\epsilon_2 [b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}] + \alpha_2 [b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}] [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]$$

$$= \delta Z \frac{d}{dZ} [b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}]$$

$$= \delta Z [\sum_{n=1}^{\infty} (-n) b_{-n} Z^{-n-1}] \qquad (2.46)$$

And lastly substituting equations (2.9), (2.43) and (2.44) in the last equation of (2.2) we get,

$$-\epsilon_1 [c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n}] + \alpha_3 [c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n}] [a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]$$

$$= \delta Z \frac{d}{dZ} [c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n}]$$

$$= \delta Z \left[\sum_{n=1}^{\infty} (-n)\, c_{-n} Z^{-n-1} \right] \tag{2.47}$$

Equating the coefficients of like powers of Z we obtain from equation (2.45)

$$\epsilon_1 a_o - \alpha_1 a_o^2 - \beta_1 a_o b_o - \gamma_1 c_o a_o = 0 \tag{2.48}$$

From equation (2.46) we have,

$$-\epsilon_2 b_o + \alpha_2 b_o a_o = 0 \tag{2.49}$$

And from equation (2.47),

$$-\epsilon_3 c_o + \alpha_3 a_o c_o = 0 \tag{2.50}$$

Thus, from equation (2.49) and (2.50) we get,

$$a_o = \frac{\epsilon_2}{\alpha_2} = \frac{\epsilon_3}{\alpha_3} \tag{2.51}$$

b_o and c_o are given by equation (2.48).

$$\epsilon_1 - \alpha_1 a_o - \beta_1 b_o - \gamma_1 c_o = 0$$

or $$\beta_1 b_o + \gamma_1 c_o = \frac{\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1}{\alpha_2}$$

Here we put the value of a_o from equation (2.51). So,

$$b_o = \frac{\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1 - \gamma_1 \alpha_2 c_o}{\beta_1 \alpha_2} \tag{2.52}$$

and

$$c_o = \frac{\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1 - \alpha_2 \beta_1 b_o}{\alpha_2 \gamma_1} \tag{2.53}$$

b_o and c_o are determined by the equation

$$\frac{b_o^{\alpha_3}}{c_o^{\alpha_2}} = A \tag{2.54}$$

where A is a constant determined by initial conditions.

Now reverting to the variable t we have the following asymptotic behaviour for the populations $N_1(t)$, $N_2(t)$ and $N_3(t)$ as $t \to \infty$.

$$\lim_{t \to \infty} N_1(t) = a_o = \frac{\epsilon_2}{\alpha_2} = \frac{\epsilon_3}{\alpha_3}$$

$$\lim_{t \to \infty} N_2(t) = b_o = \frac{\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1 - \gamma_1 \alpha_2 c_o}{\beta_1 \alpha_2} \tag{2.55}$$

$$\lim_{t \to \infty} N_3(t) = c_o = \frac{\epsilon_1 \alpha_2 - \epsilon_2 \alpha_1 - \alpha_2 \beta_1 b_o}{\alpha_2 \gamma_1}$$

Thus, all the three populations tend to constant values asymptotically. However, whereas N_1 necessarily tends to ϵ_2/α_2, the others tend to constants which are determined by the initial conditions.

SECTION - II

TWO PREY-ONE PREDATOR SYSTEM

In this section we consider the two prey-one predator system. Let the prey populations be denoted by N_1 and N_2 and the predator population by N_3. The system under consideration is described by the following set of equations :

$$\begin{aligned} \dot{N}_1 &= \epsilon_1 N_1 - \gamma_1 N_1 N_3 \\ \dot{N}_2 &= \epsilon_2 N_2 - \gamma_2 N_2 N_3 \\ \dot{N}_3 &= -\epsilon_3 N_3 + \alpha_3 N_1 N_3 + \beta_3 N_2 N_3 + \gamma_3 N_3^2 \end{aligned} \tag{3.1}$$

where the parameters ϵ_1, γ_1, ϵ_2, γ_2, ϵ_3, α_3, β_3 and γ_3 are positive and the dots on the N's signify the respective time derivatives. In terms of variable Z defined by,

$$Z = e^{\delta t},$$

where $\delta > 0$, the above equation become,

$$\begin{aligned} \delta Z \frac{dN_1}{dZ} &= \epsilon_1 N_1 - \gamma_1 N_1 N_3 \\ \delta Z \frac{dN_2}{dZ} &= \epsilon_2 N_2 - \gamma_2 N_2 N_3 \\ \delta Z \frac{dN_3}{dZ} &= -\epsilon_3 N_3 - \gamma_3 N_3^2 + \alpha_3 N_1 N_3 + \beta_3 N_2 N_3 \end{aligned} \tag{3.2}$$

From the first two equations of (3.2), equating the values of N_3 in the similar way as in the previous section we get,

$$\frac{1}{\gamma_1}\left[\delta Z \frac{1}{N_1}\frac{dN_1}{dZ} - \varepsilon_1\right] = \frac{1}{\gamma_2}\left[\delta Z \frac{1}{N_2}\frac{dN_2}{dZ} - \varepsilon_2\right]$$

or $$\gamma_2 \frac{dN_1}{N_1} - \gamma_1 \frac{dN_2}{N_2} = \frac{j}{\delta}\frac{dZ}{Z}$$

which on integration leads to,

$$\frac{N_1^{\gamma_2}}{N_2^{\gamma_1}} = B\, Z^{(j/\delta)} \qquad (3.3)$$

where B is a constant determined by the initial conditions and

$$j = \gamma_2\varepsilon_1 - \gamma_1\varepsilon_2 \qquad (3.4)$$

In view of the self interaction term present in the last equation of (3.2) which generally leads to frictional damping and saturation, we look for a solution of the system of equations such that N_3 ---> constant as t ---> ∞ , or in view of the positivity of , we look for,

$$\lim_{Z \to \infty} N_3(Z) = c_o \qquad (3.5)$$

Thus around $Z = \infty$, we have the following Laurent expansion for $N_3(Z)$:

$$N_3(Z) = c_o + \sum_{n=1}^{\infty} c_{-n} Z^{-n} \tag{3.6}$$

We then have,

$$\lim_{Z \to \infty} Z \frac{d}{dZ} (\log N_3) = 0 \tag{3.7}$$

The last equation of (3.2) can be written as,

$$\delta Z \frac{d}{dZ} (\log N_3) = -\varepsilon_3 - \gamma_3 N_3 + \alpha_3 N_1 + \beta_3 N_2$$

Using equation (3.7) we get,

$$\lim_{Z \to \infty} \alpha_3 N_1 + \beta_3 N_2 = \varepsilon_3 + \gamma_3 c_o \tag{3.8}$$

$$\equiv D$$

where D is a constant greater than ε_3.

Thus, the Laurent expansions of $N_1(Z)$ and $N_2(Z)$ around $Z = \infty$ should be,

$$N_1(Z) = a_o + \beta_3 f(Z) + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \tag{3.9}$$

$$N_2(Z) = b_o - \alpha_3 f(Z) + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \tag{3.10}$$

where f(Z) will be again a polynomial in Z with some leading power Z^m, where $m>0$. In view of the fact that populations should always be positive, i.e., $N_1(Z), N_2(Z) > 0$ for all $Z > 0$, we must have f(Z) identically equal to zero. Thus we should have the expansions for $N_1(Z)$ and $N_2(Z)$ as,

$$N_1(Z) = a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \tag{3.11}$$

$$N_2(Z) = b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \tag{3.12}$$

Substituting (3.11) and (3.12) in equation (3.3) we get,

$$\frac{[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}]^{\gamma_2}}{[b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}]^{\gamma_1}} = B\, Z^{j/\delta} \tag{3.13}$$

Three cases now arise corresponding to $j > 0$, $j < 0$ and $j = 0$.

CASE - I:- When $j > 0$.

With the same argument as in the previous section for this case, as $Z \longrightarrow \infty$ we put $b_o = 0$ and get the following asymptotic expansions for $N_1(Z)$ and $N_2(Z)$:

$$N_1(Z) = a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \tag{3.14}$$

$$N_2(Z) = \sum_{n=h}^{\infty} b_{-n} Z^{-n} \tag{3.15}$$

We now substitute equations (3.6), (3.14) and (3.15) in all the three equations of (3.2) respectively and equating coefficients of like power of Z as in the previous section, we obtain,

$$c_o = \frac{\varepsilon_1}{\gamma_1} \tag{3.16}$$

and,

$$a_o = \frac{\varepsilon_1 \gamma_3 + \varepsilon_3 \gamma_1}{\alpha_3 \gamma_1} \tag{3.17}$$

Constraint (3.3) then yields,

$$\delta = \frac{j}{h \gamma_1} \tag{3.18}$$

Reverting to the variable t we obtain the following asymptotic behaviour for the populations $N_1(t)$, $N_2(t)$ and $N_3(t)$:

$$\lim_{t \to \infty} N_1(t) = a_o = \frac{\varepsilon_1 \gamma_3 + \varepsilon_3 \gamma_1}{\alpha_3 \gamma_1}$$

$$\lim_{t \to \infty} N_2(t) = b_{-h} e^{-(j/\gamma_1)t} \longrightarrow 0 \tag{3.19}$$

$$\lim_{t \to \infty} N_3(t) = c_o = \varepsilon_1 / \gamma_1$$

Thus in this system we find that the predator population N_3 uniquely goes to the value $(\varepsilon_1 / \gamma_1)$ as $t \to \infty$, whereas one of the prey populations, N_1, tends to the value $\frac{\varepsilon_1 \gamma_3 + \varepsilon_3 \gamma_1}{\alpha_3 \gamma_1}$ and N_2 vanishes exponentially.

The constant b_{-h} is determined through

$$\frac{a_o^{\gamma_2}}{(b_{-h})^{\gamma_1}} = B \qquad (3.20)$$

where B is a constant appearing in equation (3.3) and is determined by the initial conditions.

CASE - II:- When $j < 0$

As $j < 0$, the right hand side of equation (3.13) tends to zero for $Z \to \infty$. So in this case we have $a_o = 0$, and then the asymptotic expansions for $N_1(Z)$ and $N_2(Z)$ should be,

$$N_1(Z) = \sum_{n=s}^{\infty} a_{-n} Z^{-n} \qquad (3.21)$$

$$N_2(Z) = b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \qquad (3.22)$$

whereas, the expansion for $N_3(2)$ remains the same as in equation (3.6).

Substituting equations (3.6), (3.21) and (3.22) in all the three equations of (3.2) respectively, and equating coefficients of like powers of Z, we obtain,

$$c_o = \frac{\epsilon_2}{\gamma_2} \qquad (3.23)$$

and

$$b_o = \frac{\varepsilon_3 \gamma_2 + \gamma_3 \varepsilon_2}{\beta_3 \gamma_2} \qquad (3.24)$$

constraint (3.3) yields,

$$\delta = - \frac{j}{s\gamma_2} \qquad (3.25)$$

Reverting to the variable t we obtain the following asymptotic behaviour for the populations $N_1(t)$, $N_2(t)$ and $N_3(t)$:

$$\lim_{t \to \infty} N_1(t) = a_{-s} \, e^{(j/\gamma_2)t} \longrightarrow 0$$

$$\lim_{t \to \infty} N_2(t) = b_o = \frac{\varepsilon_3 \gamma_2 + \gamma_3 \varepsilon_2}{\beta_3 \gamma_2} \qquad (3.26)$$

$$\lim_{t \to \infty} N_3(t) = c_o = \frac{\varepsilon_2}{\gamma_2}$$

Thus in this case we again find that the predator population N_3 uniquely goes to the value (ε_2 / γ_2) as $t \to \infty$, in which case one of the prey populations, N_2, tends to the value $\frac{\varepsilon_3 \gamma_2 + \gamma_3 \varepsilon_2}{\beta_3 \gamma_2}$ and N_1 vanishes exponentially. The constant a_{-s} is determined through,

$$\frac{(a_{-s})^{\gamma_2}}{b_o^{\gamma_1}} = B \qquad (3.27)$$

where, B is a constant determined by initial conditions.

CASE - III: - When j = 0

In this case for $Z \longrightarrow \infty$, equation (3.13) reduces to,

$$\frac{\left[a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n}\right]^{\gamma_2}}{\left[b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n}\right]^{\gamma_1}} = B \qquad (3.28)$$

which emplies the following Laurent expansions for $N_1(Z)$ and $N_2(Z)$:

$$N_1(Z) = a_o + \sum_{n=1}^{\infty} a_{-n} Z^{-n} \qquad (3.29)$$

$$N_2(Z) = b_o + \sum_{n=1}^{\infty} b_{-n} Z^{-n} \qquad (3.30)$$

whereas the expansion for $N_3(Z)$ is as usual as in equation (3.6). Substituting equations (3.6), (3.29) and (3.30) in the equations of (3.2) respectively and equating coefficients of like powers of Z we obtain,

$$c_o = \frac{\varepsilon_1}{\gamma_1} = \frac{\varepsilon_2}{\gamma_2} \qquad (3.31)$$

$$a_o = \frac{\varepsilon_3 \gamma_1 + \gamma_3 \varepsilon_1 - \gamma_1 \beta_3 b_o}{\alpha_3 \gamma_1} \qquad (3.32)$$

and

$$b_o = \frac{\varepsilon_3 \gamma_1 + \gamma_3 \varepsilon_1 - \gamma_1 \alpha_3 a_o}{\beta_3 \gamma_1} \qquad (3.33)$$

Now reverting to the variable t we get the following asymptotic behaviour for the populations $N_1(t)$, $N_2(t)$ and $N_3(t)$:

$$\lim_{t \to \infty} N_1(t) = a_o = \frac{\varepsilon_3 \gamma_1 + \gamma_3 \varepsilon_1 - \gamma_1 \beta_3 b_o}{\alpha_3 \gamma_1}$$

$$\lim_{t \to \infty} N_2(t) = b_o = \frac{\varepsilon_3 \gamma_1 + \gamma_3 \varepsilon_1 - \gamma_1 \varepsilon_3 a_o}{\beta_3 \gamma_1} \qquad (3.34)$$

$$\lim_{t \to \infty} N_3(t) = \frac{\varepsilon_1}{\gamma_1} = \frac{\varepsilon_2}{\gamma_2}$$

Thus all the populations tend to constant values asymptotically. However, whereas N_3 necessarily tends (ε_1 / γ_1) the others tend to constant values which are determined by the initial conditions.

The results of the present chapter are all summarised for convenience in Tables I and II.

TABLE - I

MODEL: ONE PREY-TWO PREDATOR SYSTEM	BEHAVIOUR for $t \longrightarrow \infty$
$\dot{N}_1 = \varepsilon_1 N_1 - \alpha_1 N_1^2 - \beta_1 N_1 N_2 - \gamma_1 N_1 N_3$	CASE I : $k > 0$
$\dot{N}_2 = -\varepsilon_2 N_2 + \alpha_2 N_2 N_1$	$N_1 = a_o = \varepsilon_2 / \alpha_2$
$\dot{N}_3 = -\varepsilon_3 N_3 + \alpha_3 N_3 N_1$	$N_2 = b_o = (\varepsilon_1 \alpha_2 - \varepsilon_2 \alpha_1) / \beta_1 \alpha_2$
	$N_3 = c_{-i} e^{-(k/\alpha_2)t} \longrightarrow 0$
Constraint:	CASE II : $k < 0$
$\dfrac{N_2^{\alpha_3}}{N_3^{\alpha_2}} = A Z^{k/\delta}$	$N_1 = a_o = \varepsilon_3 / \alpha_3$
	$N_2 = b_{-q} e^{(k/\alpha_3)t} \longrightarrow 0$
where	$N_3 = c_o = (\varepsilon_1 \alpha_3 - \varepsilon_3 \alpha_3) / \alpha_3 \gamma_1$
$k = \alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2$	CASE III : $k = 0$
	$N_1 = a_o = \varepsilon_2 / \alpha_2 = \varepsilon_3 / \alpha_3$
	$N_2 = b_o = \dfrac{\varepsilon_1 \alpha_2 - \varepsilon_2 \alpha_1 - \gamma_1 \alpha_2 c_o}{\beta_1 \alpha_2}$
	$N_3 = c_o = \dfrac{\varepsilon_1 \alpha_2 - \varepsilon_2 \alpha_1 - \alpha_2 \beta_1 b_o}{\alpha_2 \gamma_1}$

TABLE - II

MODEL: TWO PREY-ONE PREDATOR SYSTEM	BEHAVIOUR for $t \longrightarrow \infty$
$\dot{N}_1 = \varepsilon_1 N_1 - \gamma_1 N_1 N_3$	CASE I : $j > 0$
$\dot{N}_2 = \varepsilon_2 N_2 - \gamma_2 N_2 N_3$	$N_1 = a_o = (\varepsilon_1 \gamma_3 + \varepsilon_3 \gamma_1)/\alpha_3 \gamma_1$
$\dot{N}_3 = -\varepsilon_3 N_3 + \alpha_3 N_1 N_3 + \beta_3 N_2 N_3 - \gamma_3 N_3^2$	$N_2 = b_{-h} e^{-(j/\gamma_1)t} \longrightarrow 0$
	$N_3 = c_o = (\varepsilon_1 / \gamma_1)$
Constraint:	CASE II: $j < 0$
$\dfrac{N_1^{\gamma_2/\gamma_1}}{N_2} = BZ^{j/\delta}$	$N_1 = a_{-s} e^{(j/\gamma_2)t} \longrightarrow 0$
	$N_2 = b_o = (\varepsilon_3 \gamma_2 + \gamma_3 \varepsilon_2)/(\beta_3 \gamma_2)$
where,	$N_3 = c_o = \varepsilon_2 / \gamma_2$
$j = \gamma_2 \varepsilon_1 - \gamma_1 \varepsilon_2$	
	CASE III: $j = 0$
	$N_1 = a_o = \dfrac{\varepsilon_3 \gamma_1 + \gamma_3 \varepsilon_1 - \gamma_1 \beta_3 b_o}{\alpha_3 \gamma_1}$
	$N_2 = b_o = \dfrac{\varepsilon_3 \gamma_1 + \gamma_3 \varepsilon_1 - \gamma_1 \varepsilon_3 a_o}{\beta_3 \gamma_1}$
	$N_3 = (\varepsilon_1 / \gamma_1) = (\varepsilon_2 / \gamma_2)$

CHAPTER IV

ILLUSTRATION OF THE ANALYTICAL RESULTS USING RUNGE-KUTTA APPROXIMATION METHOD

In this chapter we illustrate our previously obtained results using the Runge-Kutta approximation method for numerical analysis. This work has been performed on the H.P. 9836A Computer. The program used or the purpose is a standard Runge-Kutta fifth order method modified by Merson (see appendix). We fed our specific numerical inputs in the program and the results under different conditions were plotted.

The purpose of the Runge-Kutta method is to obtain an approximate numerical solution of a system of first order differential equations. We discuss here the derivation of a Runge-Kuta second order method, on the basis of which higher order methods can be derived.

Runge-Kutta method is an algorithm designed to approximate the Taylor's series solutions. Let us for example consider the following system of differential equation,

$$\frac{dy_i}{dx} = y_i' = f_i(x, y_i) \qquad (4.1)$$

where, i = 1, 2, 3, ..., n.

With the initial condition, at $x = x_o$

$$y_i = y_i(x_o) \tag{4.2}$$

We seek the values, $y_i(x_o + h)$; where h is an increment of the independent variable x.

Expanding y_i about x_o in Taylor's series, we have,

$$y_i(x_o+h) = y_i(x_o) + h\, y_i'(x_o) + \frac{h^2}{2!} y_i''(x_o) + \ldots \tag{4.3}$$

We know the first derivatives,

$$y_i'(x_o) = f_i[x_o, y_i(x_o)] \tag{4.4}$$

The total differential dy_i' is written as,

$$\frac{dy_i'(x_o)}{dx} = \frac{\partial f_i[x_o, y_i(x_o)]}{\partial x} + \frac{\partial f_i[x_o, y_i(x_o)]}{\partial y_k} \frac{dy_k}{dx}$$

or,

$$\frac{dy_i'(x_o)}{dx} = y_i''(x_o) = \frac{\partial f_i[x_o, y_i(x_o)]}{\partial x} + \frac{\partial f_i[x_o, y_i(x_o)]}{\partial y_k} . f_k[x_o, y_k(x_o)] \tag{4.5}$$

where $\frac{dy_k}{dx}$ is replaced by $f_k[x_o, y_k(x_o)]$ and $k = 1, 2, 3, \ldots, n$.

Putting the values of equations (4.4) and (4.5) in equation (4.3) we get,

$$y_i(x_o+h) \doteq y_i(x_o) + hf_i[x_o, y_i(x_o)] +$$

$$\frac{h^2}{2!}\left[\frac{\partial f_i[x_o, y_i(x_o)]}{\partial x} + \frac{\partial f_i[x_o, y_i(x_o)]}{\partial y_k} \cdot f_k[x_o, y_k(x_o)]\right] \quad (4.6)$$

Equation (4.3) can also be written as,

$$y_i(x_o+h) - y_i(x_o) = \int_{x_o}^{x_o+h} f_i(x, y_i)dx \quad (4.7)$$

According to the mean value theorem there exists an x such that for

$$x = x_o + h, \quad 0 < \theta < 1$$

We have,

$$y_i(x_o+h) - y_i(x) = \int_{x_o}^{x_o+h} f_i(x, y_i)\, dx$$

$$= hf_i[x_o + \theta h, y_i(x_o + \theta h)]$$

or

$$y_i(x_o+h) = y_i(x_o) + ha_1 f_i[x_o, y_i(x_o)] +$$

$$ha_2 f_i\ [x_o + p_2 h, y_i(x_o) + q_{21} h] + \ldots \quad (4.8)$$

Here, a_1, a_2, p_2 and q_{21} are so determined that if the right hand side of equation (4.8) were expanded in power of the spacing h, the coefficients of a certain number of the leading terms would agree with the corresponding coefficients in equation (4.3).

To avoid the higher Taylor series terms evaluation we express q_{21} as a linear combination of the preceeding value of f_i. Thus, we have the approximation of the form

$$y_i(x_o+h) = y_i(x_o) + a_1k_{1i} + a_2K_{2i} \qquad (4.9)$$

where,

$$\begin{aligned} k_{1i} &= hf_i[x_o, y_i(x_o)] \\ k_{2i} &= hf_i[x_o+p_2h, y_i(x_o) + q_{21}K_{1i}] \end{aligned} \qquad (4.10)$$

Now for equation (4.6) to contain similar terms as in equation (4.9), K_{2i} must be expressed in terms of

$$f_i[x_o, y_i(x_o)], \quad \frac{\partial f_i[x_o, y_i(x_o)]}{\partial x} \quad \text{and}$$

$$\frac{\partial f_i[x_o, y_i(x_o)]}{\partial y_k} . f_k[x_o, y_k(x_o)] .$$

This can be done by expanding K_{2i} in a Taylor series for function of two variables about x_o and $y_i(x_o)$. Thus,

$$f_i[x_o+p_2h, y_i(x_o) + q_{21}K_{1i}] = f_i[x_o, y_i(x_o)] +$$

$$p_2h \left[\frac{\partial f_i[x_o, y_i(x_o)]}{\partial x}\right] + q_{21}K_{1i} \left[\frac{\partial f_i[x_o, y_i(x_o)]}{\partial y_k}\right] + \ldots$$

$$= f_i[x_o, y_i(x_o)] + p_2 h \left[\frac{\partial f_i[x_o, y_i(x_o)]}{\partial x} \right]$$

$$+ q_{21} h \left[\frac{\partial f_i[x_o, y_i(x_o)]}{\partial y_k} \right] . f_k[x_o, y_k(x_o)] + \ldots \qquad (4.11)$$

Substituting the first equation of (4.10) and (4.11) in equation (4.9) we get,

$$y_i(x_o+h) = y_i(x_o) + a_1 h f_i[x_o, y_i(x_o)] + a_2 h f_i[x_o, y_i(x_o)] + a_2 h^2 \left[\left\{ p_2 \frac{\partial f_i[x_o, y_i(x_o)]}{\partial x} \right\} + q_{21} \left\{ \frac{\partial f_i[x_o, y_i(x_o)]}{\partial y_k} \right\} . f_k[x_o, y_k(x_o)] \right\} \right] + \ldots \qquad (4.12)$$

Equating the coefficients of similar terms from equations (4.6) and (4.12) we get the following set of equations

$$a_1 + a_2 = 1$$

$$a_2 p_2 = 1/2 \qquad (4.13)$$

$$a_2 q_{21} = 1/2$$

The above set contains four unknown constants. By arbitrarily assigning a value to one unknown and then solving for the other three, we can obtain as many different sets of values as we desire and in turn as many different sets of equations (4.9) and (4.10) as desired.

For enample, if we choose $a_1 = 1/2$ in (4.13) then,

$$a_2 = 1/2$$
$$p_2 = 1 \qquad (4.14)$$
$$q_{21} = 1$$

So our equation (4.9) takes the form

$$y_i(x_o+h) = y_i(x_o) + 1/2(K_{1i} + K_{2i}) \qquad (4.15)$$

with

$$K_{1i} = hf_i[x_o, y_i(x_o)]$$
$$K_{2i} = hf_i[x_o+h, y_i(x_o) + K_{1i}] \qquad (4.16)$$

These sets of equations may be used to solve the system of first order differential equations. In this method we require two evaluations of the first derivatives in order to obtain agreement with the Taylor series solutions through terms of order h^2. A solution obtained by the use of equation (4.9) in a step-by-step integration will have a per step truncation error of order h^3, since terms containing h^3 and higher powers of h were neglected in the derivation.

By generalising the above method one can derive the Runge-Kutta fifth order method. In our case we used the standard Runge-Kutta fifth order method modified by Merson. By this method we get the accuracy and minimum step size as desired. The computation is performed a first time using step size $h_1 = h$.

The computation is again repeated, this time using step size h_2 = (h/2). Comparing these two values give an indication of the size of the error. If these two values are not sufficiently close the step size is dicreased and the same procedure is repeated till such time we get the desired accuracy.

The numerical results for models described in the previous chapter, under different conditions; are as below :

RESULTS

ONE PREY-TWO PREDATOR SYSTEM :

CASE I: For K > 0, i.e. $(\alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2) > 0$

Initial values of the populations :

$N_1(0) = 80$

$N_2(0) = 70$

$N_3(0) = 60$

Numerical inputs for different parameters :

$\varepsilon_1 = 0.12$	$\beta_1 = 0.11$
$\varepsilon_2 = 0.045$	$\gamma_1 = 0.0049$
$\varepsilon_3 = 0.0019$	$\alpha_2 = 0.0039$
$\alpha_1 = 0.0014$	$\alpha_3 = 0.015$.

The situation for this case is represented by FIG. 1.

CASE II: For K < 0, i.e., $(\alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2) < 0$.

Initial values of the populations

$N_1(0) = 80$

$N_2(0) = 70$

$N_3(0) = 60$

Numerical inputs for differnt parameters

$\varepsilon_1 = 0.12$ $\beta_1 = 0.11$

$\varepsilon_2 = 0.045$ $\gamma_1 = 0.0049$

$\varepsilon_3 = 0.0019$ $\alpha_2 = 0.0039$

$\alpha_1 = 0.0014$ $\alpha_3 = 0.015$

The situation for this case is represented by FIG. 2.

CASE III: For K = 0, i.e. $(\alpha_2 \varepsilon_3 - \alpha_3 \varepsilon_2) = 0$

Initial values of the populations

$N_1(0) = 80$

$N_2(0) = 70$

$N_3(0) = 60$

Numerical inputs for different parameters

$\varepsilon_1 = 0.12$ $\beta_1 = 0.15$

$\varepsilon_2 = 0.045$ $\gamma_1 = 0.005$

$\varepsilon_3 = 0.0019$ $\alpha_2 = 0.0039$

$\alpha_1 = 0.004$ $\alpha_3 = 0.015$

The situation for this case is represented by FIG. 3.

TWO PREY-ONE PREDATOR SYSTEM :

CASE I : - For $j > 0$, i.e. $(\gamma_2 \epsilon_1 - \gamma_1 \epsilon_2) > 0$

Initial values of the populations

$N_1(0) = 90$

$N_2(0) = 60$

$N_3(0) = 40$.

Numerical inputs for different parameters :

$\epsilon_1 = 0.18$	$\gamma_2 = 0.0032$
$\epsilon_2 = 0.0049$	$\gamma_3 = 0.002$
$\epsilon_3 = 0.09$	$\alpha_3 = 0.0019$
$\gamma_1 = 0.0021$	$\beta_3 = 0.17$

The situation is represented by FIG. 4.

CASE II: For $j < 0$, i.e. $(\gamma_2 \epsilon_1 - \gamma_1 \epsilon_2) < 0$.

Initial values of the populations :

$N_1(0) = 90$

$N_2(0) = 60$

$N_3(0) = 40$.

Numerical inputs for different parameters :

$\epsilon_1 = 0.18$	$\gamma_2 = 0.0032$
$\epsilon_2 = 0.0049$	$\gamma_3 = 0.0023$
$\epsilon_3 = 0.18$	$\alpha_3 = 0.0014$
$\gamma_1 = 0.0021$	$\beta_3 = 0.12$

The situation is represented by FIG. 5.

CASE III: For $j = 0$, i.e. $(\gamma_2 \varepsilon_1 - \gamma_1 \varepsilon_2) = 0$.

Initial values of the populations :

$N_1(0) = 90$

$N_2(0) = 60$

$N_3(0) = 40$.

Numerical inputs for different parameters :

ε_1	= 0.18	γ_2	= 0.004
ε_2	= 0.0049	γ_3	= 0.005
ε_3	= 0.12	α_3	= 0.003
γ_1	= 0.0016	β_3	= 0.2

The situation is represented by FIG. 6.

CHAPTER - V

SUMMARY OF THE RESULTS

In this dissertation we have discussed three species ecosystem models within the framework of Lotka-Volterra model. We have analysed the one prey-two predator system in which the competition and self interaction terms are excluded for the predator populations and the two prey one predator system without the self interaction and competition terms for the prey populations. We have obtained the asymptotic behaviour of the component populations in these two systems as $t \to \infty$. It has been done by exploiting a constraint that exists in the subspace of the two populations. We further used Laurent series expansions in the asymptotic region in an appropriately chosen variable. The conclusions drawn from our analytical results and numerical analysis are as below.

Let us first consider the one prey-two predator system without the self interaction and competition terms for the predator populations. We observe different behaviour of the component populations under different circumstances. Let us discuss the result for the CASE I, i.e., $K > 0$, of the system. We find that the prey population N_1 goes to a finite constant value as $t \to \infty$. Also, one of the predator populations N_2 tends to a finite value whereas the other predator population N_3 vanishes exponentially. This situation is represented by FIG. 1 which has been plotted with the help of the Computer using methods of numerical analysis. The population N_3 shows steady decrease to the zero value. The populations N_1 and N_2 oscillate with decreasing amplitudes about a finite value for which they finally settle.

For the CASE II, i.e., when $K < 0$, our analytical results show that the prey population N_1 goes to a unique finite value as $t \rightarrow \infty$. This time the predator population N_3 tends to a finite value while N_2 is annihilated exponentially. Our specimen results of this situation are plotted in FIG. 2. This situation is quite similar to the above one except that here we have the annihilation of the population N_2 instead of N_3.

It is interesting to note that for the CASE III, i.e., when $K = 0$, we have all the populations remaining finite and non vanishing i.e., there is coexistence of all the three populations. This situation is represented by FIG. 3. It is seen that all the populations show oscillations with decreasing amplitudes about a finite value for which they finally settle. These values are not all independent of initial conditions. Repeating the calculations with changed inputs we find that this general trend persists.

Next we have considered the two prey-one predator system with the exclusion of competition and self-interaction terms for the prey populations. When we take into account CASE I, i.e., when $j > 0$, we find that the predator population N_3 goes to a unique constant value as $t \to \infty$. Also one of the prey populations, N_1, tends to a constant value and the other, N_2, vanishes exponentially. This situation is presented in FIG. 4. For CASE II, i.e., when $j < 0$, we find that the predator population N_3 goes to a unique constant value as $t \to \infty$. Also, the prey populations N_2 tends to a constant value and N_1 dies out. This situation is plotted in FIG. 5. Finally, for CASE III, i.e, when $j = 0$, we obtain all the three populations to have finite and non-vanishing values, i.e., there is co-existence of all populations. This situation is shown in FIG. 6. After a few initial fluctuations, all the three populations reach certain finite constant values. These values are not independent of initial conditions. However, the same pattern is repeated for different initial conditions.

The method we have used is quite simple and can be used whenever there exists a constraint in the subspace of two populations of the interacting species.

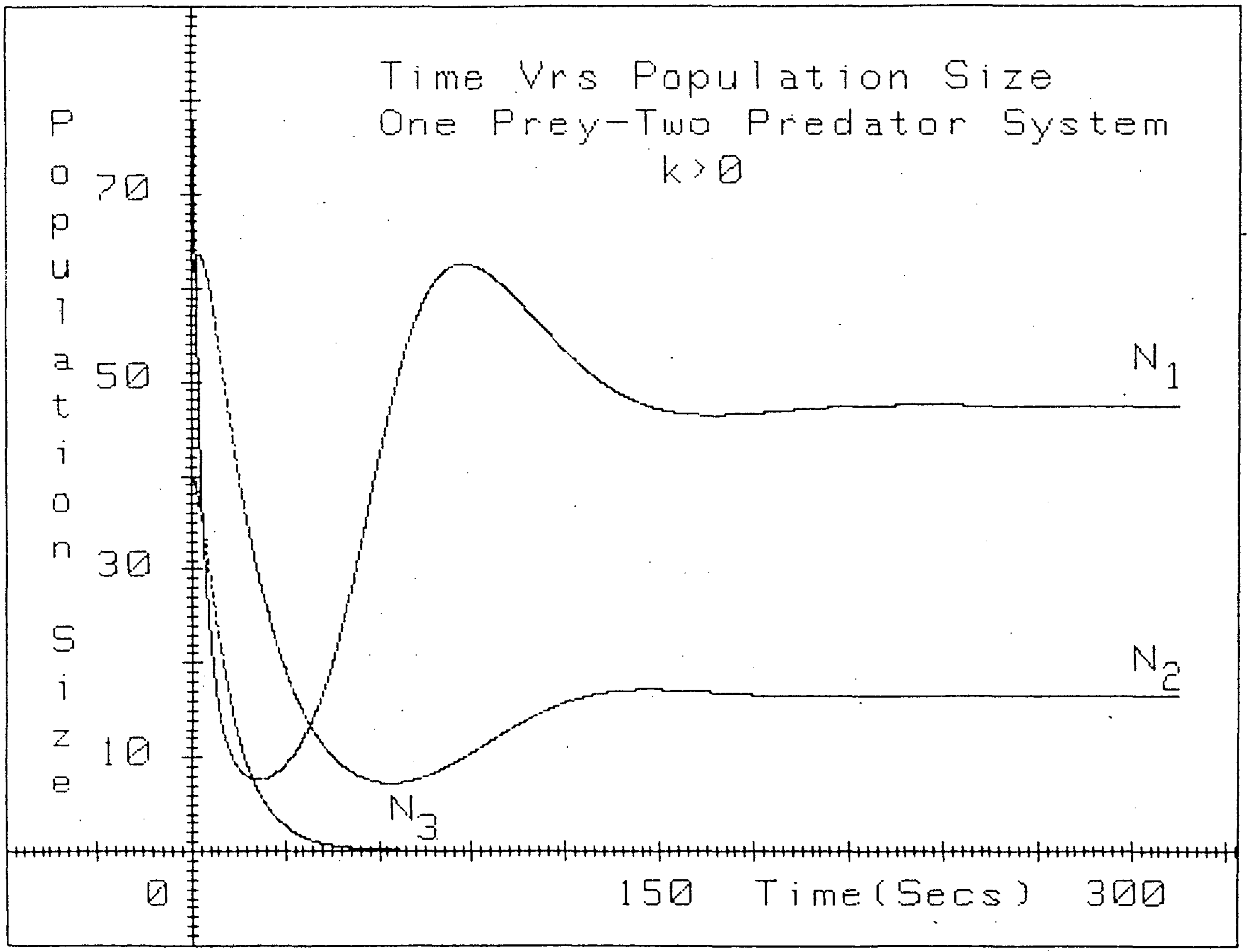

Time Vrs Population Size
One Prey-Two Predator System
k>0
Population Size
70
50
30
10
0
150
Time(Secs)
300
N1
N2
N3

FIG-1

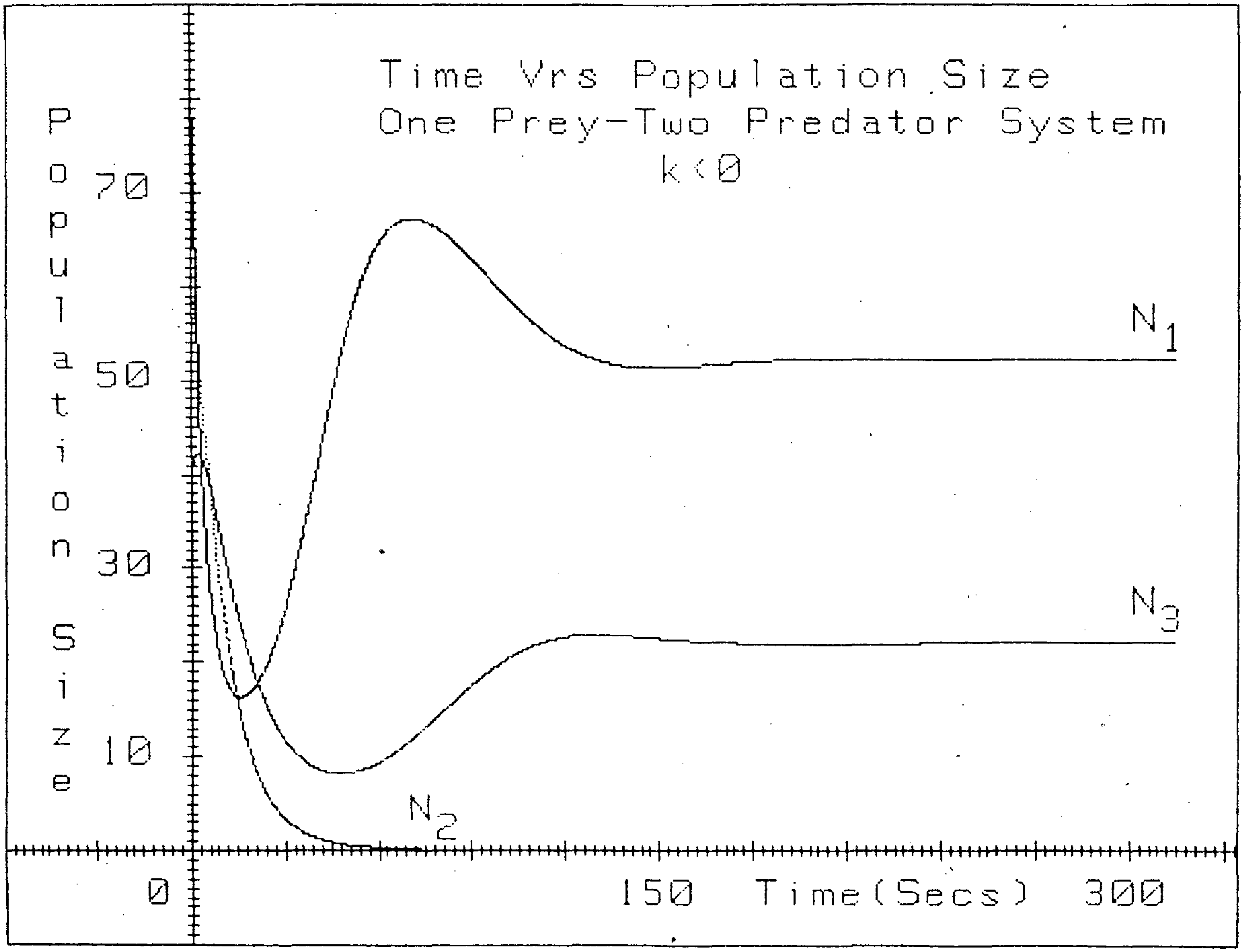

Time Vrs Population Size
One Prey-Two Predator System
k<0
Population Size
70
50
30
10
0
150
Time(Secs)
300
N1
N3
N2

FIG-2

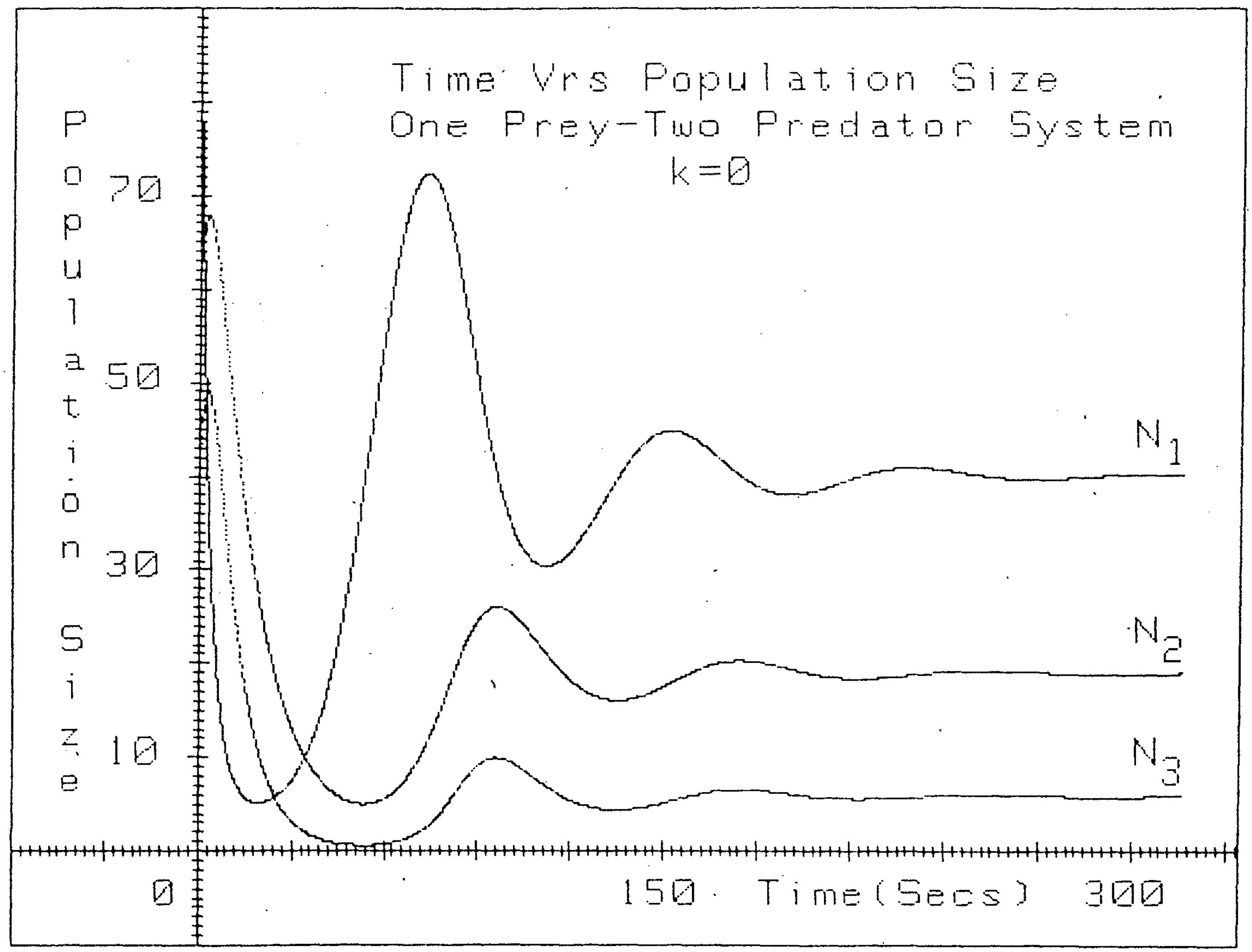

Time Vrs Population Size
One Prey-Two Predator System
k=0
Population Size
70
50
30
10
0
150
Time(Secs)
300
N1
N2
N3

FIG-3

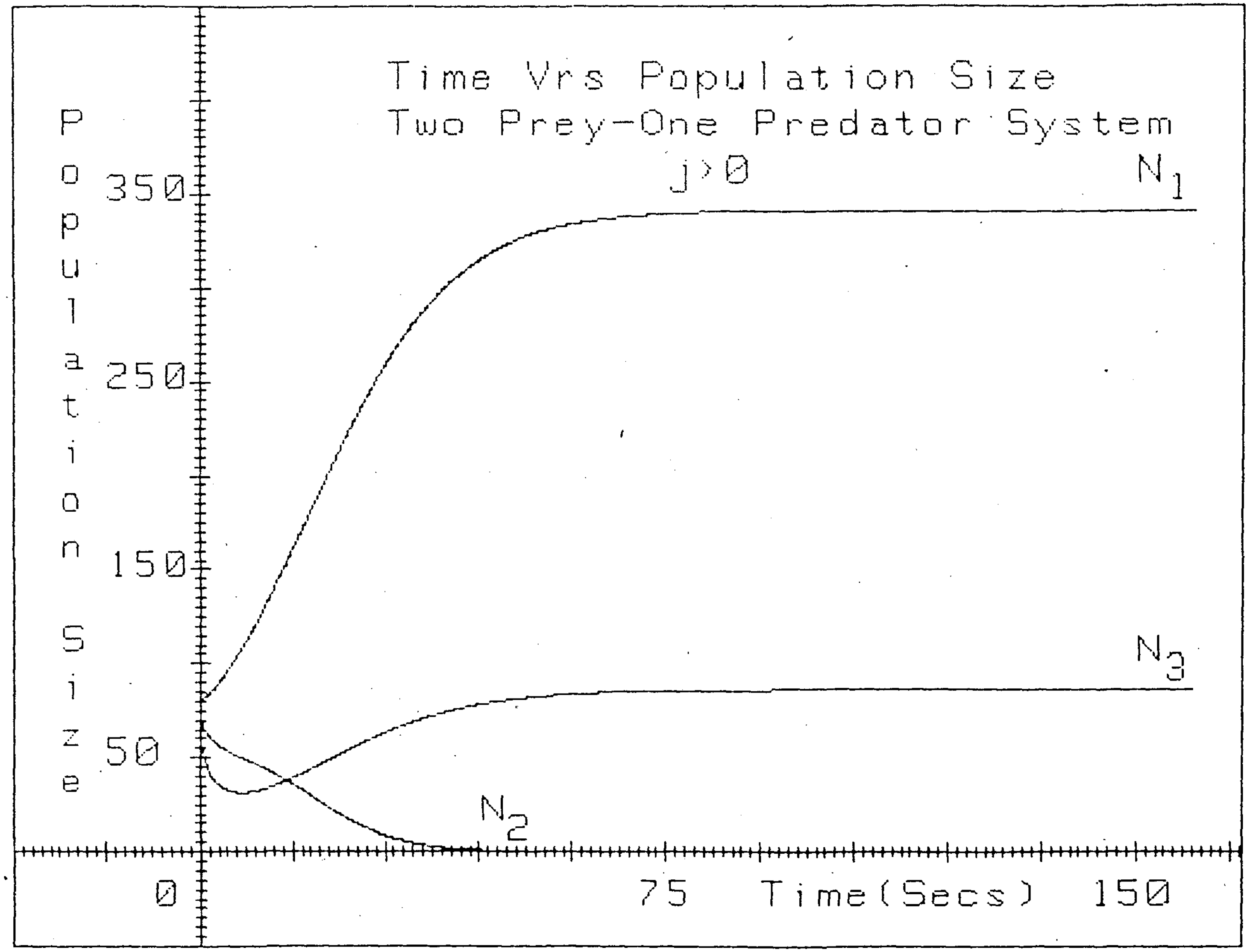
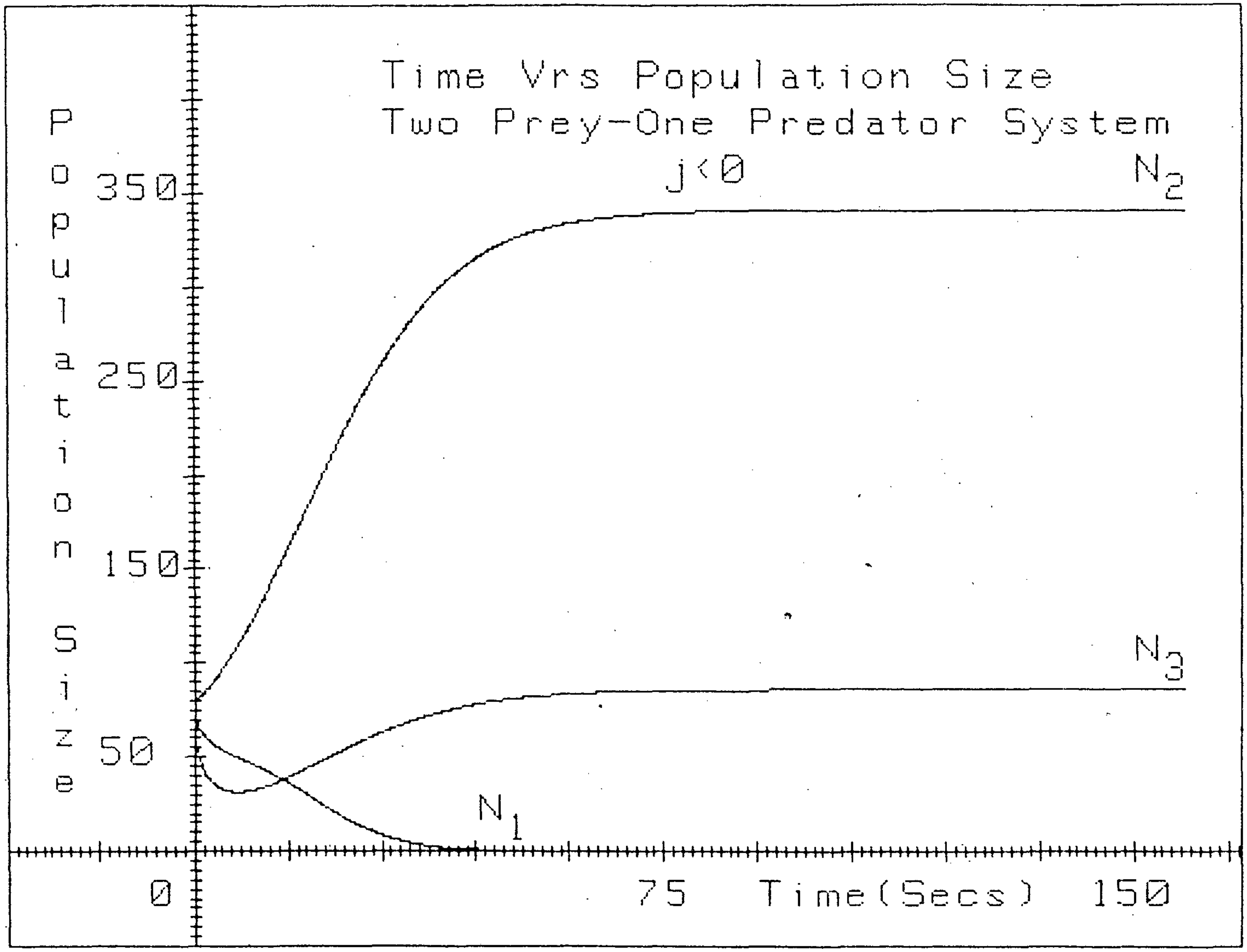

Time Vrs Population Size
Two Prey-One Predator System
j>0
N_1
N_3
N_2
Population Size
350
250
150
50
0
75
Time(Secs)
150

FIG-4



Time Vrs Population Size
Two Prey-One Predator System
j<0
N2
N3
N1
Population Size
350
250
150
50
0
75
Time(Secs)
150

FIG-5

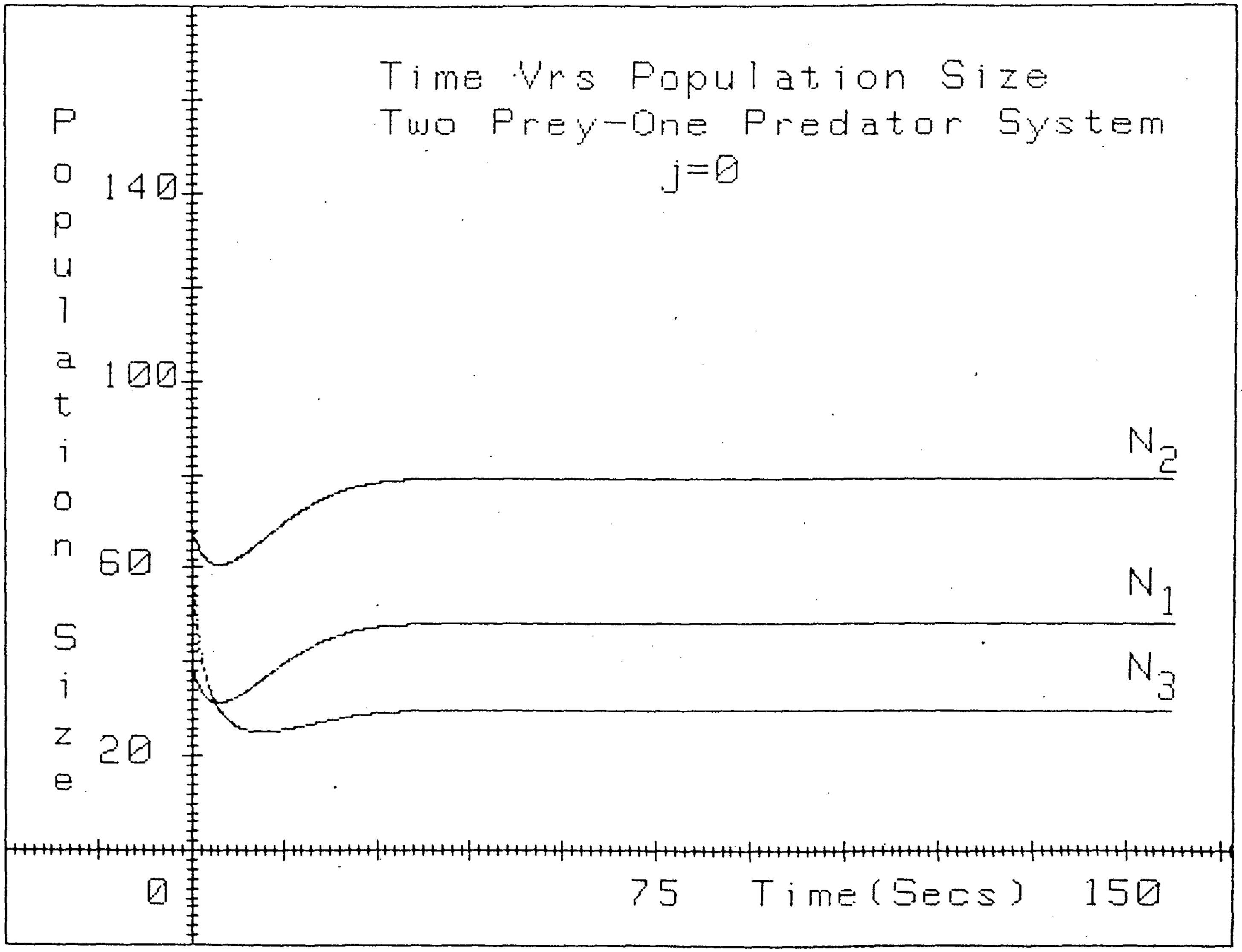

Time Vrs Population Size
Two Prey-One Predator System
j=0
Population Size
140
100
60
20
0
75
Time(Secs)
150
N2
N1
N3

FIG-6

APPENDIX I

```
10    REM RUNGA KUTTA METHOD MODIFIED BY MERSON
20    REM ONE PREY TWO PREDATOR SYSTEM
30    REM k>0
40    DIM W(10),Ak(10),Bk(10),X0(10),X(10),X1(10),X2(10),X3(10)
50    DIM X4(10),X5(10),F(10)
60    INPUT "DIMENSION OF DIFF.EQ.",Nn
70    COM A,B,C,D,Ee,Ff,G,D1
80    A=.18
90    B=.0021
100   C=.0049
110   D=.0032
120   Ee=.09
130   Ff=.0019
140   G=.17
150   D1=.002
160   READ Hh,Tin,Tend,Hprint
170   DATA .1,0,1000,.1
180   FOR I=1 TO Nn
190   READ X0(I)
200   NEXT I
210   DATA 90,60,40
220   J=0
230   Acc1=1.E-4
240   Acc2=1.E-6
250   Hmin=1.E-7
260 ! ALPHA OFF
270   GINIT
280   GRAPHICS ON
290   FRAME
300   AXES 1,1,20,10,10,10
310   MOVE 40,90
320   LABEL "Time Vrs Population Size"
330   MOVE 40,85
340   LABEL "One Prey-Two Predator System"
350   MOVE 70,80
360   LABEL "k>0"
370   MOVE 5,85
380   Label$="Population Size"
390   FOR I=1 TO 15
400   LABEL Label$[I,I]
410   NEXT I
420   MOVE 10,78
430   LABEL "70"
440   MOVE 10,58
450   LABEL "50"
460   MOVE 10,38
470   LABEL "30"
480   MOVE 10,18
490   LABEL "10"
500   MOVE 80,3
510   LABEL "Time(Secs)"
520   MOVE 15,3
530   LABEL "0"
540   MOVE 65,3
550   LABEL "150"
560   MOVE 115,3
570   LABEL "300"
580   MOVE 123,58
590   LABEL "1"
600   MOVE 123,26
610   LABEL "2"
620   MOVE 44,10
630   LABEL "3"
640   MOVE 120,60
650   LABEL "N"
660   MOVE 120,28
670   LABEL "N"
680   MOVE 41,12
690   LABEL "N"
700   MOVE 5,90
710   PRINT "INITIAL X= ";X0(1)
```

```
720     MOVE 5,87
730     PRINT "INITIAL Y= ";X0(2)
740     MOVE 5,84
750     PRINT "INITIAL Z= ";X0(3)
760     MOVE 5,80
770     PRINT " A, B, C,  D, Ee,"
780     MOVE 5,77
790     PRINT A;B;C;D;Ee
800     MOVE 5,74
810     PRINT "Ff, G , D1"
820     MOVE 2,71
830     PRINT Ff;G;D1
840     FOR I=1 TO Nn
850     W(I)=X0(I)
860     NEXT I
870     Tout=Tin+30
880     J=J+1
890     IF J=1 THEN 1020
900     MOVE (Tf/2)+5,Zf
910     Zf=X(1)+10
920     LINE TYPE 1
930     DRAW (Tout/2)+5,Zf
940     MOVE (Tf/2)+5,Xh
950     Xh=X(2)+10
960     LINE TYPE 3
970     DRAW (Tout/2)+5,Xh
980     MOVE (Tf/2)+5,Xn
990     Xn=X(3)+10
1000    LINE TYPE 8
1010    DRAW (Tout/2)+5,Xn
1020    Tf=Tout
1030    IF Tout<Tend THEN 1050
1040    STOP
1050    T=Tout
1060    Tout=Tout+Hprint
1070    Rzero=1.E-7
1080    S=Hh
1090    Iswh=0
1100    Hsv=S
1110    Cof=Tout-T
1120    IF ABS(S)<ABS(Cof) THEN 1160
1130    S=Cof
1140    IF ABS(Cof/Hsv)<Rzero THEN 1750
1150    Iswh=1
1160    FOR I=1 TO Nn
1170    X0(I)=W(I)
1180    NEXT I
1190    Ht=S*1./3.
1200    T=T+Ht
1210    CALL Gunc(X0(*),Nn,F(*))
1220    FOR I=1 TO Nn
1230    X1(I)=Ht*F(I)
1240    NEXT I
1250    FOR I=1 TO Nn
1260    X(I)=W(I)+X1(I)
1270    NEXT I
1280    CALL Gunc(X(*),Nn,F(*))
1290    FOR I=1 TO Nn
1300    X2(I)=Ht*F(I)
1310    NEXT I
1320    FOR I=1 TO Nn
1330    X(I)=W(I)+(X1(I)+X2(I))/2.
1340    NEXT I
1350    T=T+.5*Ht
1360    CALL Gunc(X(*),Nn,F(*))
1370    FOR I=1 TO Nn
1380    X3(I)=Ht*F(I)
1390    NEXT I
1400    FOR I=1 TO Nn
1410    X(I)=W(I)+.375*X1(I)+1.125*X3(I)
1420    NEXT I
1430    T=T+.5*S
```

```
1440   CALL Gunc(X(*),Nn,F(*))
1450   FOR I=1 TO Nn
1460   X4(I)=Ht*F(I)
1470   NEXT I
1480   FOR I=1 TO Nn
1490   X(I)=6.*X4(I)+1.5*X1(I)-4.5*X3(I)+W(I)
1500   NEXT I
1510   CALL Gunc(X(*),Nn,F(*))
1520   FOR I=1 TO Nn
1530   X5(I)=Ht*F(I)
1540   NEXT I
1550   FOR I=1 TO Nn
1560   X(I)=.5*X5(I)+2.*X4(I)+.5*X1(I)+W(I)
1570   NEXT I
1580   FOR I=1 TO Nn
1590   W(I)=X(I)
1600   NEXT I
1610   FOR I=1 TO Nn
1620   Ak(I)=ABS(.5*Acc1*W(I))+Acc2
1630   Bk(I)=ABS(-.5*X5(I)-4.5*X3(I)+4.*X4(I)+X1(I))
1640   NEXT I
1650   FOR I=1 TO Nn
1660   IF ABS(W(I))<Rzero THEN 1680
1670   IF Bk(I)>Ak(I) THEN 1770
1680   NEXT I
1690   IF Iswh=1 THEN 1750
1700   FOR I=1 TO Nn
1710   IF Bk(I)>.03125*Ak(I) THEN 1100
1720   NEXT I
1730   S=S*1.5
1740   GOTO 1100
1750   Hh=Hsv
1760   GOTO 1900
1770   Cof=.5*S
1780   IF ABS(Cof)>=Hmin THEN 1830
1790   S=Hmin
1800   IF Hsv<0. THEN LET S=-S
1810   IF Iswh=1 THEN 1750
1820   GOTO 1100
1830   FOR I=1 TO Nn
1840   W(I)=X0(I)
1850   NEXT I
1860   T=T-S
1870   S=Cof
1880   Iswh=0
1890   GOTO 1100
1900   GOTO 880
1910   STOP
1920   END
1930   SUB Gunc(X(*),Nn,F(*))
1940   COM A,B,C,D,Ee,Ff,G,D1
1950 F:  F(1)=A*X(1)-B*X(1)*X(1)-C*X(1)*X(2)-D*X(1)*X(3)
1960     F(2)=-Ee*X(2)+Ff*X(2)*X(1)
1970     F(3)=-G*X(3)+D1*X(1)*X(3)
1980     SUBEND
```

APPENDIX II

```
10      REM RUNGA KUTTA METHOD MODIFIED BY MERSON
20      REM TWO PREY ONE PREDATOR SYSTEM
30      REM j>0
40      DIM W(10),Ak(10),Bk(10),X0(10),X(10),X1(10),X2(10),X3(10)
50      DIM X4(10),X5(10),F(10)
60      INPUT "DIMENSION OF DIFF.EQ.",Nn
70      COM A,B,C,D,Ee,Ff,G,D1
80      A=.12
90      B=.0014
100     C=.11
110     D=.0049
120     Ee=.045
130     Ff=.0039
140     G=.0019
150     D1=.015
160     READ Hh,Tin,Tend,Hprint
170     DATA .1,0,1000,.1
180     FOR I=1 TO Nn
190     READ X0(I)
200     NEXT I
210     DATA 80,70,60
220     J=0
230     Acc1=1.E-4
240     Acc2=1.E-6
250     Hmin=1.E-7
260     ALPHA OFF
270     GINIT
280     GRAPHICS ON
290     FRAME
300     AXES 1,1,20,10,10,10
310     MOVE 40,90
320     LABEL "Time Vrs Population Size"
330     MOVE 40,85
340     LABEL "Two Prey-One Predator System"
350     MOVE 70,80
360     LABEL "j>0"
370     MOVE 5,85
380     Label$="Population Size"
390     FOR I=1 TO 15
400     LABEL Label$[I,I]
410     NEXT I
420     MOVE 80,3
430     LABEL "Time(Secs)"
440     MOVE 10,78
450     LABEL "350"
460     MOVE 10,58
470     LABEL "250"
480     MOVE 10,38
490     LABEL "150"
500     MOVE 10,18
510     LABEL "50"
520     MOVE 15,3
530     LABEL "0"
540     MOVE 67,3
550     LABEL "75"
560     MOVE 115,3
570     LABEL "150"
580     MOVE 123,78
590     LABEL "1"
600     MOVE 123,27
610     LABEL "3"
620     MOVE 53,10
630     LABEL "2"
640     MOVE 120,80
650     LABEL "N"
660     MOVE 120,29
670     LABEL "N"
680     MOVE 50,12
690     LABEL "N"
700     MOVE 5,90
710     PRINT "INITIAL X= ";X0(1)
```

```
720    MOVE 5,87
730    PRINT "INITIAL Y= ";X0(2)
740    MOVE 5,84
750    PRINT "INITIAL Z= ";X0(3)
760    MOVE 5,80
770    PRINT " A, B, C,  D, Ee,"
780    MOVE 5,77
790    PRINT A;B;C;D;Ee
800    MOVE 5,74
810    PRINT "Ff, G , D1"
820    MOVE 2,71
830    PRINT Ff;G;D1
840    FOR I=1 TO Nn
850    W(I)=X0(I)
860    NEXT I
870    Tout=Tin+30
880    J=J+1
890    IF J=1 THEN 1020
900    MOVE (Tf/1)-10,Zf
910    Zf=(X(1)/5)+10
920    LINE TYPE 1
930    DRAW (Tout/1)-10,Zf
940    MOVE (Tf/1)-10,Xh
950    Xh=(X(2)/5)+10
960    LINE TYPE 3
970    DRAW (Tout/1)-10,Xh
980    MOVE (Tf/1)-10,Xn
990    Xn=(X(3)/5)+10
1000   LINE TYPE 8
1010   DRAW (Tout/1)-10,Xn
1020   Tf=Tout
1030   IF Tout<Tend THEN 1050
1040   STOP
1050   T=Tout
1060   Tout=Tout+Hprint
1070   Rzero=1.E-7
1080   S=Hh
1090   Iswh=0
1100   Hsv=S
1110   Cof=Tout-T
1120   IF ABS(S)<ABS(Cof) THEN 1160
1130   S=Cof
1140   IF ABS(Cof/Hsv)<Rzero THEN 1750
1150   Iswh=1
1160   FOR I=1 TO Nn
1170   X0(I)=W(I)
1180   NEXT I
1190   Ht=S*1./3.
1200   T=T+Ht
1210   CALL Gunc(X0(*),Nn,F(*))
1220   FOR I=1 TO Nn
1230   X1(I)=Ht*F(I)
1240   NEXT I
1250   FOR I=1 TO Nn
1260   X(I)=W(I)+X1(I)
1270   NEXT I
1280   CALL Gunc(X(*),Nn,F(*))
1290   FOR I=1 TO Nn
1300   X2(I)=Ht*F(I)
1310   NEXT I
1320   FOR I=1 TO Nn
1330   X(I)=W(I)+(X1(I)+X2(I))/2.
1340   NEXT I
1350   T=T+.5*Ht
1360   CALL Gunc(X(*),Nn,F(*))
1370   FOR I=1 TO Nn
1380   X3(I)=Ht*F(I)
1390   NEXT I
1400   FOR I=1 TO Nn
1410   X(I)=W(I)+.375*X1(I)+1.125*X3(I)
1420   NEXT I
1430   T=T+.5*S
```

```
1440   CALL Gunc(X(*),Nn,F(*))
1450   FOR I=1 TO Nn
1460   X4(I)=Ht*F(I)
1470   NEXT I
1480   FOR I=1 TO Nn
1490   X(I)=6.*X4(I)+1.5*X1(I)-4.5*X3(I)+W(I)
1500   NEXT I
1510   CALL Gunc(X(*),Nn,F(*))
1520   FOR I=1 TO Nn
1530   X5(I)=Ht*F(I)
1540   NEXT I
1550   FOR I=1 TO Nn
1560   X(I)=.5*X5(I)+2.*X4(I)+.5*X1(I)+W(I)
1570   NEXT I
1580   FOR I=1 TO Nn
1590   W(I)=X(I)
1600   NEXT I
1610   FOR I=1 TO Nn
1620   Ak(I)=ABS(.5*Acc1*W(I))+Acc2
1630   Bk(I)=ABS(-.5*X5(I)-4.5*X3(I)+4.*X4(I)+X1(I))
1640   NEXT I
1650   FOR I=1 TO Nn
1660    IF ABS(W(I))<Rzero THEN 1680
1670    IF Bk(I)>Ak(I) THEN 1770
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1700   FOR I=1 TO Nn
1710    IF Bk(I)>.03125*Ak(I) THEN 1100
1720   NEXT I
1730   S=S*1.5
1740   GOTO 1100
1750   Hh=Hsv
1760   GOTO 1900
1770   Cof=.5*S
1780    IF ABS(Cof)>=Hmin THEN 1830
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1800    IF Hsv<0. THEN LET S=-S
1810    IF Iswh=1 THEN 1750
1820   GOTO 1100
1830   FOR I=1 TO Nn
1840   W(I)=X0(I)
1850   NEXT I
1860   T=T-S
1870   S=Cof
1880    Iswh=0
1890   GOTO 1100
1900   GOTO 880
1910   STOP
1920   END
1930   SUB Gunc(X(*),Nn,F(*))
1940   COM A,B,C,D,Ee,Ff,G,D1
1950 F:  F(1)=A*X(1)-B*X(1)*X(3)
1960     F(2)=C*X(2)-D*X(2)*X(3)
1970     F(3)=-Ee*X(3)+Ff*X(1)*X(3)+G*X(2)*X(3)-D1*X(3)*X(3)
1980     SUBEND
```

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