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Dynamics of Distal Actions on Certain Compact Spaces

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Dedication

TO MY MOTHER AND LATE FATHER

Declaration

The work presented in this thesis to best of my knowledge and belief is original except as acknowledged in the text. I, Alok Kumar Yadav, hereby declare that I have not submitted this material, either in full or in part, for a degree at this or any other institution.

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Contents

Chapter 1

Introduction

The main objective of this thesis is to study the *distality* of certain actions on real and p-adic unit spheres. We also study the distality of actions of automorphisms on $\mathrm{Sub_G}$, where G is a locally compact group and $\mathrm{Sub_G}$ is the compact space of all closed subgroups of G endowed with the Chabauty topology.

Distality was first introduced by David Hilbert (cf. Moore [23]) and further studied by many in different settings (see Abels [1, 2], Ellis [14], Furstenberg [16], Jaworksi-Raja [20], Raja-Shah [27, 28] and Shah [29]).

Let X be a (Hausdorff) topological space. Recall that, a semigroup $\mathfrak{S}$ of homeomorphisms of X is said to act *distally* on X if for every pair of distinct elements $x, y \in X$, the closure of $\{(T(x), T(y)) \mid T \in \mathfrak{S}\}$ does not intersect the diagonal $\{(d, d) \mid d \in X\}$; equivalently we say that the $\mathfrak{S}$-action on X is distal. Let $T : X \to X$ be a homeomorphism. The map T is said to be *distal* if the group $\{T^n\}_{n\in\mathbb{Z}}$ acts distally on X. If X is compact, then T is distal if and only if the semigroup

$\{T^n\}_{n\in\mathbb{N}}$ acts distally (cf. Berglund et al. [6]). Note that a homeomorphism T of a topological space is distal if and only if T^n is distal, for any $n \in \mathbb{Z}$. We now give some examples of distal maps.

Example 1.0.1. *Let $X = \mathbb{S}^1$ and let $T : \mathbb{S}^1 \to \mathbb{S}^1$ be any rotation map. Then T is distal.*

Example 1.0.2. *Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Consider $T : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ such that $T(x, y) = (x, xy)$. One can easily check that T is distal.*

Example 1.0.3. *Let $X = \mathbb{R}$ with the usual topology and consider an affine map $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = ax+b$, $a, b \in \mathbb{R}$. Then T is distal if and only if $|a| = 1$.*

In Chapter 2, we first state some known results which are useful. The action of $GL(n+1,\mathbb{R})$ on $\mathbb{S}^n$ is defined as follows: for $T \in GL(n+1,\mathbb{R})$ and $x \in \mathbb{S}^n$, $\overline{T}(x) = T(x)/\|T(x)\|$. This is a continuous group action. We prove following main result.

Theorem. (Theorem 2.2.1) *Let $\mathfrak{S} \subset GL(n+1,\mathbb{R})$ be a semigroup. Then the following are equivalent:*

- (a) *$\mathfrak{S}$ acts distally on $\mathbb{S}^n$.*
- (b) *The closure of $\mathfrak{S}\mathcal{D}/\mathcal{D}$ in $GL(n+1,\mathbb{R})$ is a compact group, where $\mathcal{D}$ is the centre of $GL(n+1,\mathbb{R})$.*
- (c) *For the semigroup $\mathfrak{S}' = \{\alpha_T T \mid T \in GL(n+1,\mathbb{R})$ and $\alpha_T = |\det T|^{-1/(n+1)}\}$, the closure of $\mathfrak{S}'$ is a compact group.*

Corollary. (Corollary 2.2.5) *Let $\mathfrak{S}$ be a closed semigroup in $GL(n+1,\mathbb{R})$ such that for every $T \in \mathfrak{S}$, $\det T = \pm 1$. Then $\mathfrak{S}$ acts distally on $\mathbb{S}^n$ if and only if every cyclic semigroup of $\mathfrak{S}$ acts distally on $\mathbb{S}^n$.*

In Chapter 3, we study the dynamics of 'affine' maps $\overline{T}_a$ on $\mathbb{S}^n$. For $T \in GL(n+1, \mathbb{R})$ and $a \in \mathbb{R} \setminus \{0\}$ such that $\|T^{-1}(a)\| \neq 1$. We define $\overline{T}_a$ on $\mathbb{S}^n$ as $\overline{T}_a(x) = \frac{a+T(x)}{\|a+T(x)\|}$. In the first section, we prove that $\overline{T}_a$ is homeomorphism if and only if $\|T^{-1}(a)\| < 1$, and discuss the existence of fixed points and periodic points of $\overline{T}_a$ on $\mathbb{S}^1$, for different choices of $T \in GL(2, \mathbb{R})$. The existence of fixed points and periodic points implies that the action of the affine map $\overline{T}_a$ on $\mathbb{S}^1$ is not distal as $\overline{T}_a^2 \neq \mathrm{Id}$. We also discuss conditions on T under which $\overline{T}_a$ on $\mathbb{S}^n$ is not distal, $n \in \mathbb{N}$. In section two of this Chapter we discuss the behaviour of fixed and periodic points of $\overline{T}_a$, for different rotations T, whether they are attracting or repelling. More precisely, we prove the following results:

Theorem. (Theorem 3.1.3) *Let $T \in GL(2, \mathbb{R})$ and let $a \in \mathbb{R}^2 \setminus \{0\}$ be such that $\|T^{-1}(a)\| < 1$. Then the following hold:*

(1) *If an eigenvalue of T is real and positive, then $\overline{T}_a$ has a fixed point.*

(2) *If the eigenvalues of T are complex of the form $r = t(\cos\theta \pm i \sin\theta)$, ($t > 0$), then $T = tABA^{-1}$ for some A in $GL(2, \mathbb{R})$ and B is a rotation by the angle θ. Suppose $\cos\theta > 0$ and $|\sin\theta| \leq \|T^{-1}(a)\|/(\|A\|\|A^{-1}\|)$. Then $\overline{T}_a$ has a fixed point.*

Proposition. (Proposition 3.1.7) *Let $T \in GL(2, \mathbb{R})$. Suppose T has either real eigen values or complex eigenvalues of the form $t(\cos\theta \pm i\sin\theta)$, ($t > 0$), where either $0 < \cos\theta < 1$ or $\|T\| > 5\sqrt{\det T}$. Then there exists an $a \in \mathbb{R}^2$, such that $0 < \|T^{-1}(a)\| < 1$ and $\overline{T}_a$ has a fixed point or a periodic point of order 2 and it is not distal.*

Theorem. (Theorem 3.1.10) *Let $T \in GL(n+1, \mathbb{R})$, $n \in \mathbb{N}$. Suppose any one of the* *following holds:*

(i) *T has two real eigenvalues or T has a complex eigen value of the form $t(\cos\theta + i\sin\theta)$, $t > 0$ such that $0 < \cos\theta < 1$.*

(ii) *T is an isometry with at least one real eigen value.*

(iii) *T is proximal with $\det T > 0$.*

Then there exists an $a \in \mathbb{R}^{n+1}$ with $0 < \|T^{-1}(a)\| < 1$ such that $\overline{T}_a$ has a fixed point or a periodic point of order 2 and $\overline{T}_a$ is not distal.

Theorem. (Theorem 3.2.1) *Let T be a rotation map and let $a \in \mathbb{R}^2 \setminus \{0\}$ be such that $\alpha = \|a\| < 1$. Then, for $\cos\theta > 0$, $|\sin\theta| < \alpha$, $\overline{T}_a$ has two fixed points; one of them is attracting and the other is repelling.*

In Chapter 4, we study the distality of the action of a semigroup (in $GL(n, \mathbb{Q}_p)$) and 'affine' actions on the p-adic unit sphere $\mathcal{S}_n = \{x \in \mathbb{Q}_p^n \mid \|x\|_p = 1\}$. Here, the action of $GL(n, \mathbb{Q}_p)$ on $\mathcal{S}_n$ is defined as follows: for $T \in GL(n, \mathbb{Q}_p)$ and $x \in \mathcal{S}_n$, $\overline{T}(x) = \|T(x)\|_p T(x)$. For semigroups of $SL(n, \mathbb{Q}_p)$, we prove a result analogous to Theorem 2.2.1 (see Theorem 4.1.4). We shall then define for $T \in GL(n, \mathbb{Q}_p)$ and some nonzero a an 'affine' action $\overline{T}_a : \mathcal{S}_n \to \mathcal{S}_n$ as follows $\overline{T}_a(x) = \|a + T(x)\|_p \, (a + T(x))$, $x \in \mathcal{S}_n$. We discuss the dynamics of $\overline{T}_a$, which is different from the real case. This Chapter contains the following main results:

Proposition. (Proposition 4.1.3) Let $T \in GL(n, \mathbb{Q}_p)$. If bT is distal for some $b \in \mathbb{Q}_p$, then $\overline{T}$ is distal. Conversely, if $\overline{T}$ is distal, then for some $m \in \mathbb{N}$ and $l \in \mathbb{Z}$, $p^l T^m$ is distal. If $|\det T|_p = 1$ and $\overline{T}$ is distal, then T is distal.

Theorem. (Theorem 4.1.4) Let $\mathfrak{S} \subset SL(n, \mathbb{Q}_p)$ be a semigroup. Then the following are equivalent:

1. $\mathfrak{S}$ acts distally on $\mathcal{S}_n$.
2. The group generated by $\mathfrak{S}$ acts distally on $\mathcal{S}_n$.
3. The closure of $\mathfrak{S}$ is a compact group.

Theorem. (Theorem 4.2.2) Suppose $T \in GL(n, \mathbb{Q}_p)$. Let $\overline{T}_a : \mathcal{S}_n \to \mathcal{S}_n$ be defined as $\overline{T}_a(x) = \|a + T(x)\|_p(a + T(x))$, $x \in \mathcal{S}_n$. There exists an open compact group V such that for all $a \in V \setminus \{0\}$ we have $\|T^{-1}(a)\|_p < 1$ and the following hold:

(I) If $\overline{T}$ is distal, then $\overline{T}_a$ is distal for all nonzero $a \in V$.

(II) If $\overline{T}$ is not distal, then for every neighbourhood U of 0 contained in V, there exists a nonzero $a \in U$ such that $\overline{T}_a$ is not distal.

In Chapter 5, we study the space $\mathrm{Sub_G}$ of all closed subgroups of a topological group G endowed with the Chabauty topology. We first study the behaviour of sequences in $\mathrm{Sub_G}$ and prove some elementary results. There is a natural action of Aut(G) on $\mathrm{Sub_G}$ which is defined as $\mathrm{Aut(G)} \times \mathrm{Sub_G} \to \mathrm{Sub_G}$, $(T, H) \mapsto T(H)$; $T \in \mathrm{Aut(G)}$, $H \in \mathrm{Sub_G}$. For $T \in \mathrm{Aut(G)}$, we study the distality of T on $\mathrm{Sub_G}$. Let G^0 denote the connected component of the identity in G. In this this section we prove the following main result:

Theorem. (Theorem 5.2.5) Let G be a locally compact metrizable group, $T \in \mathrm{Aut(G)}$ and let K be the maximal compact normal subgroup of G^0. If T is distal on $\mathrm{Sub_G}$, then T is distal on G/K^0. Moreover, if T acts distally on K^0, then T acts distally on G.

Chapter 2 and a part of Chapter 3 contain results from the work done in [30]. Chapter 4 contains results from the work done in [31]. Chapter 5 contains results from an ongoing collaboration.

Chapter 2

Dynamics of semigroup actions on compact Hausdorff spaces

In this chapter, the first section covers definitions and known results which are useful in the proof of main results in the later part of the chapter. In the second section we prove the result which characterises the distality of the action of a semigroup of $GL(n+1,\mathbb{R})$ on $\mathbb{S}^n$.

2.1 Definitions and Known results

Definition 2.1.1. *Let T be an invertible linear map on the real vector space $\mathbb{R}^n$. Define $C(T) := \{x \in \mathbb{R}^n \mid T^n(x) \to 0$ as $n \to \infty\}$. Observe that $C(T)$ is a T-invariant subspace called the* **contraction space** *of T. Observe that, for $T \in GL(n,\mathbb{R})$, $T|_{C(T)}$ has all the eigenvalues of absolute value less than one and $T|_{C(T^{-1})}$*

has all the eigenvalues of absolute value greater than one.

We list here some known results which are useful in the proof of main results.

Lemma 2.1.2. *([13] Lemma 2.1) Let V be a finite-dimensional vector space over $\mathbb{R}$. Let $\{\alpha_n\}$ be a sequence in $GL(V)$. Then there exists a subspace W of V with the property that there exists a subsequenec $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $\{\alpha_n(v)\}$ converges for all $v \in W$ and $\alpha_{n_k}(v) \to \infty$ for $v \notin W$.*

Lemma 2.1.3. *([2], Theorem $1'$) Let T be a linear transformation in $GL(n,\mathbb{C})$. Then T is distal if and only if every eigen value of T has absolute value one.*

Theorem 2.1.4. *([14], Theorem 1) If X is a compact space and T is a surjective continuous map on X, then T is distal if and only if $E(T) = \overline{\{T^n \mid n \in \mathbb{Z}\}}$ is a group.*

Theorem 2.1.5. *([15], Theorem 1.1) Suppose that the subgroup $G \subset GL(n,\mathbb{C})$ satisfies the following:*

(i) Every element of G is semisimple and all its eigenvalues have absolute value 1.

(ii) G is closed with respect to the ordinary topology of $GL(n,\mathbb{C})$. Then G is conjugate in $GL(n,\mathbb{C})$ to a subgroup of $U_n(\mathbb{C})$ and therefore compact.

Note that, for $x \in \mathbb{R}^n$ we shall consider $\|x\|$ as an euclidean distance from 0 in $\mathbb{R}^n$ throughout the dissertation. For $x \in \mathbb{R}^n \setminus \{0\}$, let $\overline{x} = x/\|x\|$.

2.2 Distal action of semigroups on $\mathbb{S}^n$

In this section, we consider a semigroup $\mathfrak{S}$ of $GL(n+1,\mathbb{R})$ and study the distality of its canonical actions on $\mathbb{S}^n$. For $T \in GL(n+1,\mathbb{R})$, let $\overline{T} : \mathbb{S}^n \to \mathbb{S}^n$ be defined as $\overline{T}(x) = T(x)/\|T(x)\|$. Note that if $T \in GL(n+1,\mathbb{R})$ is distal, it does not imply that $\overline{T}$ is distal. For example, if we consider $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. T is a distal map on $\mathbb{R}^2$ but $\overline{T}$ is not distal on $\mathbb{S}^1$, as $\overline{T}^n(x) \to (1,0)$, for all $x \in \mathbb{S}^1 \setminus \{(-1,0)\}$. Recall that $\mathfrak{S} \subset GL(n+1,\mathbb{R})$ acts on $\mathbb{S}^n$ as follows: for $T \in GL(n+1,\mathbb{R})$.

$$GL(n+1,\mathbb{R}) \times \mathbb{S}^n \to \mathbb{S}^n,\ (T,x) \mapsto \overline{T}(x), \text{ where } x \in \mathbb{S}^n.$$

It is a continuous group action. The following theorem characterises distal actions of semigroups of $GL(n+1,\mathbb{R})$ on $\mathbb{S}^n$.

Theorem 2.2.1. *Let $\mathfrak{S} \subset GL(n+1,\mathbb{R})$ be a semigroup. Then the following are equivalent:*

(a) *$\mathfrak{S}$ acts distally on $\mathbb{S}^n$.*

(b) *The closure of $\mathfrak{S}\mathcal{D}/\mathcal{D}$ in $GL(n+1,\mathbb{R})/\mathcal{D}$ is a compact group, where $\mathcal{D}$ is the centre of $GL(n+1,\mathbb{R})$.*

(c) *For the semigroup $\mathfrak{S}' = \{\alpha_T T \mid T \in GL(n+1,\mathbb{R})$ and $\alpha_T = |\det T|^{-1/(n+1)}\}$, the closure of $\mathfrak{S}'$ is a compact group.*

We first prove another result which will be required in the proof of the above theorem.

In the following proposition, for $T \in GL(n,\mathbb{C})$, where $\mathbb{C}$ is the field of complex numbers, the condition that $\overline{\{T^m\}}_{m\in\mathbb{N}}$ is compact, is equivalent to the condition that $\overline{\{T^m\}}_{m\in\mathbb{N}}$ is a compact group (cf. [19]). It is also equivalent to the condition that T is semi-simple and its eigenvalues are of absolute value one; (this fact is well-known). Therefore the following proposition shows that Theorem 1.1 of [15] holds for semigroups.

Proposition 2.2.2. *Let $\mathfrak{S}$ be a closed semigroup in $GL(n,\mathbb{C})$ such that for every element $T \in \mathfrak{S}$, the closure of $\{T^m\}_{n\in\mathbb{N}}$ in $GL(n,\mathbb{C})$ is compact. Then $\mathfrak{S}$ is a compact group.*

Proof. Let $H_T = \overline{\{T^m\}}_{m\in\mathbb{N}}$. Since H_T is compact, it is a group (see Sec. 1 in Ch. A of [19]). Therefore every element of $\mathfrak{S}$ is invertible and hence $\mathfrak{S}$ itself is a group. Moreover, as H_T is a compact group, T is semi-simple and eigenvalues of T are of absolute value one. Hence by Theorem 1.1 of [15], $\mathfrak{S}$ is contained in a conjugate of the unitary group $U_n(\mathbb{C})$. In particular, $\mathfrak{S}$ is a compact group. □

Remark 2.2.3. *In Proposition 2.2.2, we have considered a closed semigroup because there exists a subgroup of $GL(n,\mathbb{C})$ with non-compact closure such that every element of the subgroup generates a relatively compact group (see Counterexample 1.10 [5]).*

Proof of Theorem 2.2.1: $(a) \Rightarrow (c)$: Suppose the $\mathfrak{S}$-action on $\mathbb{S}^n$ is distal. Since the action of $\mathfrak{S}'$ on $\mathbb{S}^n$ is same as that of $\mathfrak{S}$, we have that the action of $\mathfrak{S}'$ on $\mathbb{S}^n$ is distal. Moreover, as the closure of $\mathfrak{S}'$ is also a semigroup whose elements have determinant ± 1, and it acts distally on $\mathbb{S}^n$, we may assume that $\mathfrak{S}'$ is closed. We first show that for every $T \in \mathfrak{S}'$, $\{T^n\}_{n\in\mathbb{N}}$ is relatively compact. Let $T \in \mathfrak{S}'$ be fixed. As $\det T = \pm 1$, at least one of the following holds: (i) all the eigenvalues of T are of absolute value one, (ii) at least one eigenvalue of T has absolute value less than one and at least one eigenvalue of T has absolute value greater than one.

If possible, suppose $\{T^m\}_{m\in\mathbb{N}}$ is not relatively compact in $\mathfrak{S}'$. Then there exists $\{m_k\} \subset \mathbb{N}$ such that $\{T^{m_k}\}$ is divergent, i.e. it has no convergent subsequence. Moreover, we show that there exist a subsequence of $\{m_k\}$, which we denote by $\{m_k\}$ again, and a nonzero vector v_0 such that $\{T^{m_k}(v_0)\}$ converges.

Suppose (i) holds. If 1 or -1 is an eigenvalue of T, then there exists a nonzero eigenvector v such that $T(v) = v$ or $T(v) = -v$. If all the eigenvalues of T are complex and of absolute value one, then there exists a two dimensional subspace W' such that $T|_{W'}$, being conjugate to a rotation map, generates a relatively compact group in $GL(W')$. Hence, for every $v_0 \in W' \setminus \{0\}$, $\{T^{m_k}(v_0)\}$ is relatively compact and has a subsequence which converges.

Now suppose (ii) holds. Then T has at least one eigenvalue of absolute value less than one. Therefore, the contraction group $C(T)$ is nontrivial, and we can choose v_0 as any nonzero vector in $C(T)$.

By Lemma 2.1 of [13] (see Lemm 2.1.2 above), there exist a subspace W of $\mathbb{R}^{n+1}$ and a subsequence $\{l_k\}$ of $\{m_k\}$ such that $\{T^{l_k}(v)\}$ converges for every $v \in W$ and $\|T^{l_k}(v)\| \to \infty$, whenever $v \notin W$. Here $W \neq \{0\}$ as $v_0 \in W$.

Now we show that $W = \mathbb{R}^{n+1}$. If possible, suppose $W \neq \mathbb{R}^{n+1}$. Then there exists $u \in \mathbb{R}^{n+1} \setminus W$ such that $\|T^{l_k}(u)\| \to \infty$. Since $\mathbb{S}^n$ is compact, passing to a subsequence if necessary, we have $\overline{T}^{l_k}(u) = T^{l_k}(u)/\|T^{l_k}(u)\| \to a$, for some $a \in \mathbb{S}^n$. Let $v_0 \in W$ be as above. As $u \notin W, u + v_0 \notin W$ and therefore $T^{l_k}(u + v_0) \to \infty$. As $\{T^{l_k}(v_0)\}$ is bounded, we get that $\overline{T}^{l_k}(u + v_0) \to a$. Here, $\overline{u} \neq \overline{u + v_0}$ as $v_0 \in W$ and $u \notin W$. This is a contradiction as $\mathfrak{G}$ acts distally on $\mathbb{S}^n$. Hence $W = \mathbb{R}^{n+1}$, and therefore $\{T^{l_k}\}$ is bounded. As $\{l_k\}$ is a subsequence of $\{m_k\}$, we arrive at a contradiction to our earlier assumption that $\{T^{m_k}\}$ is divergent. Hence $\{T^m\}_{m\in\mathbb{N}}$ is relatively compact and all its limit points belong to $\mathfrak{G}'$. Therefore, by

Proposition 2.2.2, $\mathfrak{S}'$ is a compact group.

$(c) \Rightarrow (b)$ as $\overline{\mathfrak{S}\mathcal{D}}/\mathcal{D} = \overline{\mathfrak{S}'}\mathcal{D}/\mathcal{D}$ which is a compact group. It is easy to see that $(b) \Rightarrow (a)$ as on $\mathbb{S}^n$, the action of $\mathfrak{S}$ is same as the action of $\mathfrak{S}\mathcal{D}/\mathcal{D}$ whose closure is a compact group which acts distally. $\square$

Remark 2.2.4. *From Theorem 2.2.1, it follows that for a semigroup* $\mathfrak{S} \subset GL(n+1,\mathbb{R})$ *whose all elements have determinant* 1 *or* -1*, the distality of the* $\mathfrak{S}$*-action on* $\mathbb{S}^n$ *implies the distality of the* $\mathfrak{S}$*-action on* $\mathbb{R}^{n+1}$*. In the latter part of the proof of the theorem, instead of [15], one can also use the results about the structure of distal linear groups from [1] and give a different argument.*

There are examples of actions of semigroups on compact spaces which are not distal but every cyclic subsemigroup acts distally, (see Example 2.5 [20]). However, the latter does not happen in the case of closed semigroups of $SL(n+1,\mathbb{R})$ for the action on $\mathbb{S}^n$.

Corollary 2.2.5. *For a closed semigroup* $\mathfrak{S}$ *of* $SL(n+1,\mathbb{R})$*, the following holds:* $\mathfrak{S}$ *acts distally on* $\mathbb{S}^n$ *if and only if every cyclic semigroup of* $\mathfrak{S}$ *acts distally on* $\mathbb{S}^n$*.*

Proof. The"only if" statement is obvious. Now suppose every cyclic semigroup of $\mathfrak{S}$ acts distally on $\mathbb{S}^n$. Let $T \in \mathfrak{S}$. By Theorem 2.2.1, $\overline{\{T^m\}}_{m\in\mathbb{N}}$ is a compact group. Now the assertion follows from Proposition 2.2.2. $\square$

Chapter 3

Dynamics of ‘affine’ maps on $\mathbb{S}^n$

This Chapter has two sections. In the first section, we define an ‘affine’ action on unit sphere $\mathbb{S}^n$, $n \in \mathbb{N}$, and study its dynamics. In the second section we study the behaviour of fixed points.

3.1 Dynamics of ‘affine’ actions

Consider the affine action on $\mathbb{R}^{n+1}$, $T_a(x) = a + T(x)$, where $T \in GL(n+1, \mathbb{R})$, and $a \in \mathbb{R}^{n+1}$. In this section, we first consider the corresponding ‘affine’ map $\overline{T}_a$ on $\mathbb{S}^n$ which is defined for any nonzero a satisfying $\|T^{-1}(a)\| \neq 1$ as follows: $\overline{T}_a(x) = T_a(x)/\|T_a(x)\|, x \in \mathbb{S}^n$. (For $a = 0$, $\overline{T}_a = \overline{T}$, which is studied in Chapter 2). Observe that $T_a(x) = 0$ for some $x \in \mathbb{S}^n$ if and only if $T^{-1}(a)$ has norm 1. Therefore, $\overline{T}_a$ is well defined if $\|T^{-1}(a)\| \neq 1$. The map $\overline{T}_a$ is a homeomorphism for any nonzero a satisfying $\|T^{-1}(a)\| < 1$ (see Lemma 4.2.1). In this section, we study the dynamics of such homeomorphsims $\overline{T}_a$.

Note that any nontrivial homeomorphism S of $\mathbb{S}^1$ with a fixed or a periodic point is not distal unless some power of S is an identity map. In fact if S has a fixed point or a periodic point of order 2, then either $S^2 = \mathrm{Id}$, or there exist $x, y \in \mathbb{S}^1$, $x \neq y$, such that $S^{2n}(x) \to z$ and $S^{2n}(y) \to z$ for some fixed point z of S^2. This can be seen through an identification of $\mathbb{S}^1$ to $[0,1]$ and getting an increasing homeomorphism of $[0,1]$ equivariant to S^2, as the latter is orientation preserving. These facts are well-known, we refer to [25] and [9] for more details. We will discuss the existence of fixed points or periodic points of order 2 for $\overline{T}_a$ on $\mathbb{S}^1$, $T \in GL(2,\mathbb{R})$, under certain conditions on the eigenvalues of T and the norm of T. In Lemma 3.1.2, we prove that for the homeomorphic map $\overline{T}_a$ on $\mathbb{S}^1$, $\overline{T}_a^2$ is nontrivial. Therefore, if $\overline{T}_a$ has a fixed point or a periodic point of oder 2, then $\overline{T}_a$ is not distal.

Lemma 3.1.1. *Let $T \in GL(n+1,\mathbb{R})$ and let $a \in \mathbb{R}^{n+1} \setminus \{0\}$ be such that $\|T^{-1}(a)\| \neq 1$. The map $\overline{T}_a$ on $\mathbb{S}^n$ is a homeomorphism if and only if $\|T^{-1}(a)\| < 1$.*

Proof. Suppose $\|T^{-1}(a)\| < 1$. From the definition, it is clear that $\overline{T}_a$ is continuous. It is enough to show that $\overline{T}_a$ is a bijection since any continuous bijection on a compact Hausdorff space is a homeomorphism. Suppose $x, y \in \mathbb{S}^n$ such that $\overline{T}_a(x) = \overline{T}_a(y)$. Then we have $(a + T(x))/\|a + T(x)\| = (a + T(y))/\|a + T(y)\|$ or $(1-\beta)T^{-1}(a) = \beta y - x$, where $\beta = \|a + T(x)\|/\|a + T(y)\|$. If $\beta \neq 1$, then we get that $\|T^{-1}(a)\| \geq 1$, a contradiction. Hence $\beta = 1$, and $x = y$. Therefore, $\overline{T}_a$ is injective.

Let $y \in \mathbb{S}^n$ be fixed. Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as follows: $\Psi(t) = \|tT^{-1}(y) - T^{-1}(a)\|$, $t \in \mathbb{R}^+$. Clearly, Ψ is a continuous map, and hence the image of Ψ is connected. We have $\Psi(0) = \|T^{-1}(a)\| < 1$ and $\Psi(t) \to \infty$ as $t \to \infty$. Therefore there exists a $t_0 \in \mathbb{R}^+$, $t_0 \neq 0$ such that $\Psi(t_0) = 1$. Let $x = t_0T^{-1}(y) - T^{-1}(a)$. Then

$x \in \mathbb{S}^n$ and $\overline{T}_a(x) = y$. Hence $\overline{T}_a$ is surjective.

Conversely, if $\|T^{-1}(a)\| > 1$, then $\overline{T}_a(x) = \overline{T}_a(-x) = a/\|a\|$, for $x = T^{-1}(a)/\|T^{-1}(a)\| \in \mathbb{S}^n$, i.e. $\overline{T}_a$ is not injective. $\square$

Lemma 3.1.2. *For the homeomorphism $\overline{T}_a$ on $\mathbb{S}^1$, $\overline{T}_a^2 \neq \mathrm{Id}$.*

Proof. If possible, suppose $\overline{T}_a^2 = \mathrm{Id}$. Therefore $\overline{T}_a^2(x) = x$ and $\overline{T}_a^2(-x) = -x$, for an arbitrary $x \in \mathbb{S}^1$. From the definition of $\overline{T}_a$, for $T \in GL(2, \mathbb{R})$, we have

$$b_1 a + T(a) + T^2(x) = b_2 x \tag{3.1}$$

$$b_1' a + T(a) - T^2(x) = -b_2' x \tag{3.2}$$

Where $b_1 = \|a + T(x)\|$, $b_1' = \|a - T(x)\|$, $b_2 = \|b_1 a + T(a) + T^2(x)\|$ and $b_2' = \|b_1' a + T(a) - T^2(x)\|$. Adding above equations, we get that

$$(b_1 + b_1')a + 2T(a) = (b_2 - b_2')x.$$

This implies that either x or $-x$ belongs to the positive cone generated by a and $T(a)$ in $\mathbb{R}^2$. Since x is an arbitrary element of $\mathbb{S}^1$, it shows that every element of $\mathbb{S}^1$ belongs to the positive cone generated by a and $T(a)$, which is a contradiction. Therefore $\overline{T}_a^2 \neq \mathrm{Id}$. $\square$

Observe that, $\mathbb{R}^2$ is isomorphic to the field $\mathbb{C}$ of complex numbers and $\mathbb{S}^1$ is a group under multiplication. For $x \in \mathbb{R}^2 \setminus \{0\}$, we take x^{-1} as the inverse of x in $\mathbb{C}$.

Theorem 3.1.3. *Let $T \in GL(2, \mathbb{R})$ and let $a \in \mathbb{R}^2 \setminus \{0\}$ be such that $\|T^{-1}(a)\| < 1$. Then the following hold:*

(1) *If an eigenvalue of T is real and positive, then $\overline{T}_a$ has a fixed point.*

(2) *If the eigenvalues of* T *are complex of the form* $r = t(\cos\theta \pm i\sin\theta)$, *(*$t > 0$*), then* $T = tABA^{-1}$ *for some* A *in* $GL(2,\mathbb{R})$ *and* B *is a rotation by the angle* θ. *Suppose* $\cos\theta > 0$ *and* $|\sin\theta| \leq \|T^{-1}(a)\|/(\|A\|\|A^{-1}\|)$. *Then* $\overline{T}_a$ *has a fixed point.*

Remark 3.1.4. *If* T *has complex eigenvalues, then we may assume in (2) above that* $\det A = \pm 1$ *and* A *is unique up to isometry. For if* $ABA^{-1} = CBC^{-1}$ *where* B *is a rotation and* $B \neq \pm\mathrm{Id}$, *then* $C^{-1}A$ *commutes with* B *and hence it is a rotation. Therefore,* $\|A\|\|A^{-1}\|$ *is uniquely defined for any such* T. *We will deal separately in Proposition 3.1.5 the case when* $t^{-1}T$ *is a rotation by an angle* θ *(as we may take* $A = \mathrm{Id}$ *in this particular case).*

Proof of Theorem 3.1.3: Note that $\overline{T}_a = (\overline{\beta T})_{(\beta a)}$ for all $\beta > 0$. Without loss of any generality, we can assume that $T \in GL(2,\mathbb{R})$ such that $\det T = \pm 1$. Observe that $\overline{T}_a$ has a fixed point if there exists $\gamma > 0$ such that $\gamma\mathrm{Id} - \mathrm{T}$ is invertible and $x_\gamma = (\gamma\mathrm{Id} - \mathrm{T})^{-1}(\mathrm{a})$ has norm 1. However, such a γ may not exist. Hence, we deal with some of the special cases below separately.

(1) Suppose T has a positive real eigenvalue. There exists $A \in GL(2,\mathbb{R})$ such that $T = ABA^{-1}$ where $B = \begin{bmatrix} t & 0 \\ 0 & s \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ for $t > 0$ and $ts = \pm 1$. Let $a \in \mathbb{R}^2 \setminus \{0\}$ be fixed such that $\|T^{-1}(a)\| < 1$. Let $A^{-1}(a) = (a_1, a_2)$.

Now consider $B = \begin{bmatrix} t & 0 \\ 0 & s \end{bmatrix}$. Note that if $a_2 = 0$ then $\overline{a} = a/\|a\|$ is a fixed point of $\overline{T}_a$. If $s = 1$ then $T = \mathrm{Id}$, and $\overline{a}$ is a fixed point of $\overline{T}_a$ for $a \in \mathbb{R}^2$ as above. Now let $s \neq 1$. Suppose $a_1 = 0$. If $s > 0$ then $\overline{a}$ is a fixed point of $\overline{T}_a$. Now suppose $s < 0$. Then $ts = -1$ and $t - s > 0$. Here, $\|T^{-1}(a)\| =$

$\|AB^{-1}A^{-1}(a)\| = \|A(s^{-1}A^{-1}(a))\| = \|A(0, s^{-1}a_2)\| = \|ta\| < 1$ (which is given). Therefore, $\|A(0, a_2/(t-s))\| = \|(t-s)^{-1}a\| = (t^2+1)^{-1}\|ta\| < 1$. Moreover, as A is an invertible linear map, $\|A(x, a_2/(t-s))\| \to \infty$ as $|x| \to \infty$. Therefore there exists a real number $x_0 \neq 0$ such that $\|A(x_0, a_2/(t-s))\| = 1$. It is easy to see that $x = A(x_0, a_2/(t-s))$ is a fixed point of $\overline{T}_a$.

Let a_1 and a_2 be nonzero. Consider $f : \mathbb{R} \setminus \{t, s\} \to \mathbb{R}^+$ defined by $f(\gamma) = \|A(a_1/(\gamma - t), a_2/(\gamma - s))\|$. As $\gamma \to 0$, $f(\gamma) \to \|T^{-1}(a)\| < 1$, and as $\gamma \to t_0$, $f(\gamma) \to \infty$, where $t_0 = \min\{t, s\}$ if $s > 0$, and $t_0 = t$ if $s < 0$. Therefore there exists a $\gamma_0 \in]0, t_0[$ such that $\|A(a_1/(\gamma_0 - t), a_2/(\gamma_0 - s))\| = 1$. It is easy to check that $A(a_1/(\gamma_0 - t), a_2/(\gamma_0 - s))$ is a fixed point of $\overline{T}_a$.

Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. If $a_2 = 0$, then $\overline{a}$ is a fixed point for $\overline{T}_a$. As $\|T^{-1}(a)\| < 1$, arguing as above we can find a $\gamma_1 \in]0, 1[$ and show that $A(a_1/(\gamma_1 - 1) + a_2/(\gamma_1 - 1)^2, a_2/(\gamma_1 - 1))$ has norm one and it is a fixed point of $\overline{T}_a$.

(2) As T has complex eigenvalues $t(\cos\theta \pm i \sin\theta)$, $(t > 0)$, we have that $\det T > 0$. Hence we may assume that $\det T = 1$ and $T = ABA^{-1}$, where B is a rotation by the angle θ. Let $r_1 = \cos\theta > 0$, $r_2 = \sin\theta$ and $A^{-1}(a) = (a_1, a_2)$. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $g(\gamma) = \|A(\gamma \mathrm{Id} - \mathrm{B})^{-1}\mathrm{A}^{-1}(\mathrm{a})\|$. Then $g(0) = \|T^{-1}(a)\| < 1$. As B is an isometry, we have that $\|A^{-1}(a)\| = \|A^{-1}T^{-1}(a)\|$. Here, $g(r_1) \geq |r_2|^{-1}\|A^{-1}(a)\|/\|A^{-1}\| \geq |r_2|^{-1}\|T^{-1}(a)\|/[\|A\|\|A^{-1}\|] \geq 1$. Then there exists a $\gamma_2 \in]0, r_1]$ such that $g(\gamma_2) = 1$. It is easy to check that $A(\gamma_2 \mathrm{Id} - \mathrm{B})^{-1}\mathrm{A}^{-1}(\mathrm{a}) = (\gamma_2 \mathrm{Id} - \mathrm{T})^{-1}(\mathrm{a})$ is a fixed point of $\overline{T}_a$. □

Proposition 3.1.5. *Suppose $T = B$ i.e. $T(x) = rx$, and $a \in \mathbb{R}^2 \setminus \{0\}$, where a and B are as in Theorem 3.1.3 (2). Then $\overline{T}_a(x)$ admits a fixed point if and only if*

$\cos\theta \geq \sqrt{1-\alpha^2}$ and $|\sin\theta| \leq \alpha$, where $r = \cos\theta + i\sin\theta$ and $\alpha = \|a\|$. Moreover, if fixed points of $\overline{T}_a$, for $r = (\cos\theta, \sin\theta)$ and $a = (a_1, a_2)$, exist then they are of the form

$$a(t-r)^{-1}, \text{ where } t = \cos\theta \pm \sqrt{\alpha^2 - \sin^2\theta}.$$

Proof. Suppose $\overline{T}_a$ has a fixed point, say, x_0. Then $(a + rx_0)/\|a + rx_0\| = x_0$ or $a + rx_0 = bx_0$, where $b = \|a + rx_0\|$. Let $r_1 = \cos\theta$ and $r_2 = \sin\theta$. Since $\|x_0\| = 1$, we have $a = [(b - r_1) - ir_2]\, x_0$, therefore b satisfies a quadratic equation $b^2 - 2br_1 + 1 - \alpha^2 = 0$. As $\alpha < 1$ and $b > 0$, we have that $r_1 > 0$. From the above equation we obtain $b = r_1 \pm \sqrt{r_1^2 - (1-\alpha^2)}$. As $r_1, b \in \mathbb{R}^+$, we see that $r_1 \geq \sqrt{1-\alpha^2}$ and $|r_2| \leq \alpha$.

Conversely, suppose $r_1 \geq \sqrt{1-\alpha^2}$ and $|r_2| \leq \alpha$. As $r_1 \geq \sqrt{1-\alpha^2} > 0$, $t = r_1 \pm \sqrt{r_1^2 - (1-\alpha^2)}$ are positive real numbers, for which $\|t - r\| = \alpha$. If we choose $x_t = a(t-r)^{-1}$, then $\|x_t\| = 1$ and $\overline{T}_a(x_t) = x_t$. Note that, $\overline{T}_a$ has only one fixed point if $r = \left(\sqrt{1-\alpha^2}, \pm\alpha\right)$. □

Note that for $T = -\mathrm{Id}$, a rotation by $r = (-1, 0)$ on $\mathbb{R}^2$, $\overline{T}_a$ on $\mathbb{S}^1$ has only four periodic points of order 2; namely $\overline{a}, -\overline{a}, x_0, a - x_0$, where x_0 is such that $\|x_0\| = \|a - x_0\| = 1$, which we would prove in Chapter-4 in Corollary 3.2.2 and also discuss their behaviour. The following Lemma 3.1.10 discusses the existence of periodic points of order two for $\overline{T}_a$, where T is a rotation belonging to the specific region of $\mathbb{S}^1$.

Lemma 3.1.6. *Let $T(x) = rx$, for $r = \cos\theta + i\sin\theta$, on $\mathbb{S}^1$, and a, α, $\overline{T}_a$ be as in Proposition 3.1.5.*

(*i*) *If $\cos\theta > 0$ and $|\sin\theta| > \alpha$, then $\overline{T}_a$ has no periodic point of order two.*

(*ii*) *There exists a neighbourhood U of $(-1,0)$ such that for $(\cos\theta, \sin\theta) \in U$, $\overline{T}_a$ has four periodic points of order two.*

Proof. Here $\overline{T}_a(x) = (a+rx)/b_1$ and $\overline{T}_a^2(x) = (b_1 a + ra + r^2 x)/b_2$, where $b_1 = \|a+rx\|$ and $b_2 = \|b_1 a + ra + r^2 x\|$. Observe that, $\overline{T}_{sa}^n(sx) = s\overline{T}_a^n(x)$, for $s \in \mathbb{S}^1$, $n \in \mathbb{N}$. Hence $\overline{T}_a^n(x) \to x_0$ if and only if $\overline{T}_{sa}^n(x) \to sx_0$. Therefore without loss of any generality, we can replace a by sa, where $s = (\overline{a})^{-1}(0,1)$, for $(0,1) \in \mathbb{S}^1$ and assume that $a = (0,\alpha)$.

Step I. Consider $r_1 > 0$ and $r_2 > \alpha$. Let $\psi : [0,1] \to \mathbb{S}^1$ be such that $\psi(t) = e^{2\pi i(1/4-t)}$ and $\psi(0) = \psi(1) = (0,1)$. Let $\beta = \psi^{-1}(\overline{T}_a^{-1}(r^{-1}\overline{a}))$. Then for $x \in \psi([0,\beta[)$, $\psi^{-1}(x) < \psi^{-1}(\overline{T}_a^2(x))$. For $x = (x_1, x_2) \in \psi([\beta, 1])$ such that $x_2 \le 0$, and $\overline{T}_a^2(x) = (y_1, y_2)$ such that $y_2 > 0$. This implies that $\overline{T}_a$ has no periodic point of order two.

Let $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be defined by $\varphi((x_1, x_2)) = (-x_1, x_2)$, and $\overline{T}'_a(x) = (a + r'x)/\|a + r'x\|$, where $r' = r_1 - ir_2$ and $x \in \mathbb{S}^1$. It is easy to see that $\varphi \circ \overline{T}_a = \overline{T}'_a \circ \varphi$, as φ is a linear map. Therefore, the dynamics of $\overline{T}_a$ and $\overline{T}'_a$ will be same. Now for the case of $r_1 > 0$ and $r_2 < -\alpha$, we can take $r' = r_1 - ir_2$ and $\overline{T}'_a$, and argue as above for r' and $\overline{T}'_a$, and get that $\overline{T}'_a$, and hence, $\overline{T}_a$ has no periodic point of order two.

Step II. $\overline{T}_a^2(1,0) = (x_1, x_2)$, where $x_1 = (r_1^2 - r_2^2 - r_2\alpha)/b_2$, $x_2 = (b_1\alpha + r_1\alpha + 2r_1r_2)/b_2$.

$\overline{T}_a^2(0,1) = (y_1, y_2)$, where $y_1 = (-2r_1r_2 - r_2\alpha)/b_2, y_2 = (r_1^2 - r_2^2 + r_1\alpha + b_1\alpha)/b_2$.

$\overline{T}_a^2(0,-1) = (z_1, z_2)$, where $z_1 = (2r_1r_2 - r_2\alpha)/b_2$, $z_2 = (r_2^2 - r_1^2 + r_1\alpha + b_1\alpha)/b_2$.

As $\alpha < 1$ there exists a neighbourhood U of $(-1,0)$ such that if $(r_1, r_2) \in U$ with $r_2 > 0$, $x_i > 0, y_i > 0$ and $z_i < 0$ for $i = 1,2$. Let $U = \{(x,y) \mid x \le -\sqrt{1-\epsilon^2}, |y| < \epsilon\}$ be a neighbourhood of $(-1,0)$ such that $\epsilon = min\{\lambda_1, \lambda_2, \mu_1, \mu_2, \nu\}$, where

$$\begin{aligned}
\lambda_1 &= 1/(\alpha+2) \\
\lambda_2 &= (\alpha\sqrt{1+\alpha^2}-\alpha)/2 \\
\mu_1 &= \sqrt{1-\alpha^2/4} \\
\mu_2 &= \sqrt{(1-\alpha^2)/2} \\
\nu &= \sqrt{(1-\alpha^2)/(\alpha+2)}.
\end{aligned}$$

It is easy to check that $\epsilon = \lambda_2$. Let $E = \{(x_1, x_2) \in \mathbb{S}^1 \mid x_1 \geq 0, x_2 \geq 0\}$ and $F = \{(x_1, x_2) \in \mathbb{S}^1 \mid x_1 \geq 0, x_2 \leq 0\}$. Let E^0 denote the interior of the set E.

$$\overline{T}_a^2((1,0)) \in E^0 \text{ as } r_2 < min\{\lambda_1, \lambda_2\}.$$

$$\overline{T}_a^2((0,1)) \in E^0 \text{ as } r_2 < min\{\mu_1, \mu_2\}.$$

$$\overline{T}_a^2((0,-1)) \in (F \cup E)^c \text{ as } r_2 < \nu.$$

Therefore, as $E, F \subset \mathbb{S}^1$ are compact and connected, and $\overline{T}_a^2$ is injective, we see that $\overline{T}_a^2(E) \subsetneq E$ and $F \subsetneq \overline{T}_a^2(F)$. As both E and F are homeomorphic to a closed interval in $\mathbb{R}$, considering the continuous map on the closed interval corresponding to $\overline{T}_a^2$, we see that $\overline{T}_a^2$ has fixed points in E, i.e. there exists an $x = (x_1, x_2) \in E$ such that $\overline{T}_a^2(x) = x$ and $\overline{T}_a^2(\overline{T}_a(x)) = \overline{T}_a(x)$. Similarly, there exists $y \in F$ such that y and $\overline{T}_a(y)$ are periodic points of $\overline{T}_a$ of order two.

For the rotation $r = (r_1, r_2) \in U$ with $r_2 < 0$, we can take $r' = r_1 - ir_2$ and $\overline{T}'_a$ as in previous Step I, and use the above argument for r' and $\overline{T}'_a$ to deduce that $\overline{T}'_a$, and hence $\overline{T}_a$ has four periodic points of order two. $\square$

Proposition 3.1.7. *Let $T \in GL(2, \mathbb{R})$. Suppose T has either real eigen values or a complex eigenvalue of the form $t(\cos\theta \pm i \sin\theta)$, ($t > 0$), where either $0 < \cos\theta < 1$*

or $\|T\| > 5\sqrt{\det T}$. *Then there exists an* $a \in \mathbb{R}^2$, *such that* $0 < \|T^{-1}(a)\| < 1$ *and* $\overline{T}_a$ *has a fixed point or a periodic point of order 2 and it is not distal.*

Remark 3.1.8. *Given any* T *with complex eigenvalues, we can take* $T' = CTC^{-1}$, *which has the same eigenvalues as* T *but the norm of* T' *is very large. Consider* $T = tABA^{-1}$, $t^2 = \det T$ *as above with* B *a rotation by an angle* θ. *Take* $C = C(\beta)A^{-1}$, *where* $C(\beta) = \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix}$. *Then for* $\beta > 1$, T' *has norm greater than* $\beta^2|t\sin\theta| > 5\sqrt{\det T}$ *if* $\beta > \sqrt{5}/|\sin\theta|$; *(here,* $\sin\theta \neq 0$*). That is, given any* $T \in GL(2,\mathbb{R})$, *there exist a conjugate* S *of* T *in* $GL(2,\mathbb{R})$ *and a nonzero* $a \in \mathbb{R}^2$ *such that* $\|S^{-1}(a)\| < 1$ *and* $\overline{S}_a$ *is not distal.*

Proof of Proposition 3.1.7: We may assume that $\det T = \pm 1$. If T has at least one real positive eigenvalue then by Theorem 3.1.3(1), $\overline{T}_a$ is not distal for all a satisfying $0 < \|T^{-1}(a)\| < 1$. If the eigenvalues of T are real and negative, then for the eigenvector a, $\bar{a} = a/\|a\|$ is a periodic point of order 2 for $\overline{T}_a$, and hence it is not distal.

Now suppose T has complex eigenvalues $\cos\theta \pm i\sin\theta$ with $0 < \cos\theta < 1$. Then $\det T = 1$. Suppose T is an isometry. Since $\sin\theta \neq 1$, we can choose $a \in \mathbb{R}^2$ such that $|\sin\theta| < \|a\| < 1$ and we have $\|T^{-1}(a)\| = \|a\| < 1$. Now by Theorem 3.1.3, $\overline{T}_a$ has a fixed point. If T is not an isometry, $T = ABA^{-1}$ and $\|T\| > 1$. As $0 < r_1 = \cos\theta < 1$, it is easy to check that T^2 is not an isometry and $\|T^2\| > 1$. There exists $a \neq 0$ such that $\|T^{-1}(a)\| < 1$ and $\|T(a)\| = \|T^2(T^{-1}(a))\| > 1$. Now let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as in the proof of Theorem 3.1.3. Then $0 < g(0) < 1$ and $g(2r_1) = \|T(a)\| > 1$. Therefore, there exists $\gamma_3 \in]0, 2r_1[$, such that $g(\gamma_3) = 1$ and hence $A(\gamma_3\mathrm{Id}{-}\mathrm{B})^{-1}\mathrm{A}^{-1}(\mathrm{a})$ is a fixed point for $\overline{T}_a$. In either case, $\overline{T}_a$ is not distal as $\overline{T}_a^2$ is non-trivial. Now suppose $r_1 \leq 0$. For g as above, $g(1) = (1/[2(1{-}r_1)])\|a{-}T^{-1}(a)\|$, Since $\|T\| > 5$, there exist a such that $\|T^{-1}(a)\| < 1$ and $\|a\| > 5$. Then $g(1) > (1/4)(5-1) = 1$. Then, there exists γ_4 such that $0 < \gamma_4 < 1$ and $g(\gamma_4) = 1$. Therefore, $A(\gamma_4 \mathrm{Id} - \mathrm{B})^{-1}\mathrm{A}^{-1}(\mathrm{a})$ is a fixed point for $\overline{T}_a$.

Remark 3.1.9. *If $\overline{T}_a$ on $\mathbb{S}^1$ has a fixed point or a periodic point of order 2, as mentioned earlier, $\overline{T}_a^2$ is orientation preserving and if $\overline{T}_a^2 \neq \mathrm{Id}$, then there exist $x, y, z \in \mathbb{S}^1$, $x \neq y$ such that $\overline{T}_a^{2m}(x) \to z$ and $\overline{T}_a^{2m}(y) \to z$ as $m \to \infty$ where z is a fixed point of $\overline{T}_a^2$.*

A map $T \in GL(n, \mathbb{R})$ is said to be *proximal* if it has a unique (real) eigenvalue of maximal absolute value and which has algebraic (and hence geometric) multiplicity one.

Theorem 3.1.10. *Suppose $T \in GL(n+1, \mathbb{R})$, $n \in \mathbb{N}$. Suppose any one of the following holds:*

(*i*) *T has two real eigenvalues or T has a complex eigen value of the form $t(\cos\theta + i\sin\theta)$, $t > 0$ such that $0 < \cos\theta < 1$.*

(*ii*) *T is an isometry with at least one real eigen value.*

(*iii*) *T is proximal with $\det T > 0$.*

Then there exists an $a \in \mathbb{R}^{n+1}$ with $0 < \|T^{-1}(a)\| < 1$ such that $\overline{T}_a$ has a fixed point or a periodic point of order 2 and $\overline{T}_a$ is not distal.

Proof. Suppose (i) holds. Then T keeps a 2-dimensional subspace V invariant and the restriction of T to V satisfies the condition in Proposition 3.1.7 and hence there exists $a \in V$ for which the assertion holds.

Now suppose (ii) holds. Since T has at least one real eigenvalue, it has a fixed point. If it has two real eigenvalues, the assertion will hold as above. Now suppose all except one eigenvalues of T are complex. Then T keeps a 3-dimensional space W invariant such that $T|_W$ is an isometry and has the form $\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$. Without loss of any generality we may assume that $n = 2$. Let a be such that $0 < \|a\| < 1$, $T(a) = \pm a$ and $\overline{T}_a(\overline{a} = \pm\overline{a})$. Let $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$. and $D = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Observe that U and D are also isometries, $U(a) = a$, $D(a) = \pm a$ and $T = UD = DU$. Then $\overline{T}_a = U\overline{D}_a = \overline{D}_a U$, and hence $\overline{T}_a^m = U^m \overline{D}_a^m$ for all $m \in \mathbb{N}$. Let b be an eigenvector for D other than a such that $\langle a, b\rangle = 0$. Let W' be a two dimensional space generated by a and b. Then $D(W') = W'$, D and $\overline{D}_a$ keep $W' \cap \mathbb{S}^2$ invariant, the latter is isomorphic to $\mathbb{S}^1$ as we take the same norm on W'. As $D|_W$ has an eigenvalue 1, by Theorem 3.1.3 $\overline{D}_a$ has a fixed point in $W' \cap S^2$. For $S = D|_W$, $\overline{S}_a = \overline{D}_a|_{W'\cap\mathbb{S}^2}$ and $\overline{S}_a^2$, and hence $\overline{D}_a^2$ is nontrivial on $W' \cap \mathbb{S}^2$. From Remark 11, there exist $x, y \in W'$ such that $\|x\| = \|y\| = 1$ and $\overline{D}_a^{2m}(x) \to z$ and $\overline{D}^{2m}(x) \to z$ as $m \to \infty$ for some fixed point z of $\overline{D}_a^2$. Since U is an isometry, there exists a sequence $\{m_k\}$ such that $U^{m_k} \to \mathrm{Id}$. Hence, $\overline{T}_a^{2m_k}(x) \to z$ and $\overline{T}_a^{2m_k}(x) \to z$ as $k \to \infty$. This implies that $\overline{T}_a$ is not diatal.

Suppose (iii) holds, i.e. T is proximal and $\det T > 0$. Then either T has two distinct real eigen values or the only real eigen value λ is positive. In the first case the assertion follows from (i). In the second case we have a 3-dimensional space W'' which is T-invariant. It is enough to show the assertion for $T|_{W''}$ as that is also proximal with a positive determinant. Without loss of any generality, we may assume that $n = 2$. Since λ is dominant and also positive, replacing T by T/λ, we may assume that $\lambda = 1$ and the restriction of T to a 2-dimensional subspace V_2 has eigenvalues of absolute value less than 1, i.e. $V_2 = C(T)$. Let us take $a \neq 0$ such that $T(a) = a$ with $\|a\| < 1$. Then $\overline{T}_a(\overline{a}) = \overline{a}$. Let a_0 be such that $\|a_0\| = 1$ and $\langle a_0, x\rangle = 0$ for all $x \in V_2$. Then a_0 and V_2 generate the whole space. Let $y \in V_2$ be such that $a = \alpha_0(a_0) + y$, where $\alpha_0 > 0$ as $a \notin V_2$ and $\alpha_0 \leq \|a\| < 1$. For any x in $V_2 = C(T)$ with $\|x\| = 1$, we have $\overline{T}_a(x) = (a+T(x))/\|a+T(x)\|$. Let $\alpha_1 = \|a+T(x)\| = \|\alpha_0 a_0+y+T(x)\| \geq \alpha_0$. Similarly, for $m \geq 2$, we have $\overline{T}_a^m(x) = (s_m a + T^m(x))/\|s_m a + T^m(x)\|$, where $s_1 = 1$ and $s_m = 1 + \sum_{i=1}^{m-1} \alpha_i > 1$, for $\alpha_i = \|s_i a + T^i(x)\| = \|s_i(\alpha_0 a_0 + y) + T^i(x)\| \geq \alpha_0$, for all i. Therefore, $s_m = 1 + \sum_{i=1}^{m-1} \alpha_i \geq 1 + (m-1)\alpha_0 \to \infty$ as $m \to \infty$. Moreover, $T^m(x) \to 0$ as $m \to \infty$ as $x \in C(T)$. Therefore, we get that $\overline{T}_a^m(x) = (s_m a + T^m(x))/\|s_m a + T^m(x)\| \to a/\|a\| = \overline{a}$ as $m \to \infty$. Since this holds for every $x \in V_2$ with $\|x\| = 1$, we have that $\overline{T}_a$ is not distal. In fact, if we take any point $z \in \mathbb{S}^2$ which is a positive linear combination of a and some $y \in V_2$ with $\|y\| = 1$, it is easy to show that $\overline{T}_a^m(z) \to a/\|a\|$. □

Any isometry in $GL(n+1, \mathbb{R})$ always has a real eigenvalue if $n \in \mathbb{N}$ is even. For any odd number $n \in \mathbb{N}$, any proximal map in $GL(n+1, R)$ will have two distinct real eigenvalues.

The following corollary shows that for a large class of T in $GL(n, \mathbb{R})$, there exists a nonzero $a \in \mathbb{R}^{n+1}$ such that $\overline{T}_a$ is a homeomorphism and it is not distal.

Corollary 3.1.11. *For $T \in GL(n+1, \mathbb{R})$, the following statements hold:*

1. *There exist a conjugate S of T in $GL(n+1, \mathbb{R})$ and $a \in \mathbb{R}^{n+1} \setminus \{0\}$ such that*

$\|S^{-1}(a)\| < 1$ and $\overline{S}_a$ on $\mathbb{S}^n$ is not distal.

2. *There exists $a \in \mathbb{R}^{n+1} \setminus \{0\}$ such that for some $S \in \{T, T^2, T^3\}$ such that $\|S^{-1}(a)\| < 1$ and $\overline{S}_a$ on $\mathbb{S}^n$ is not distal.*

Proof. For T as above, either T has two real eigenvalues or a complex eigenvalue of the form $t(\cos\theta + i\sin\theta)$, $t > 0$. In the first case, both the assertions follows from Theorem 3.1.10 (i) for $S = T$. In the second case suppose $0 < \cos\theta < 1$, then both the assertions follows from Theorem 3.1.10 (i) for $S = T$. Now suppose $\cos\theta \leq 0$. As T keeps a two dimensional space V invariant, we can replace T by $T|_V$ and assume that $n = 1$. Now (1) follows from Proposition 3.1.7 and the Remark 10. For the second assertion, if $\cos\theta = 0$, then T^2 has two identical real eigenvalues equal to -1, and if $\cos\theta < 0$, then either $\cos(2\theta) > 0$ or $\cos(3\theta) > 0$. In either of these cases, (2) follows for $S \in \{T, T^2, T^3\}$ from Theorem 3.1.10 (i). $\square$

3.2 Behaviour of fixed points

In the previous section, we have discussed the existence of fixed points and periodic points of $\overline{T}_a$ on $\mathbb{S}^1$ for different rotations (see Proposition 3.1.5 and Lemma 3.1.6). In this section we study the behaviour of fixed points and periodic points for different rotation map T, whether they are attracting or repelling.

Let X be a locally compact metric space and $f : X \to X$ a continuous map. A fixed point p of f is **attracting** if it has a neighbourhood U such that $\overline{U}$ is compact, $f(\overline{U}) \subset U$, and $\bigcap\limits_{n\geq 0} f^n(U) = \{p\}$. A fixed point p is **repelling** if it has a neighbourhood U such that $\overline{U} \subset f(U)$, and $\bigcap\limits_{n\geq 0} f^{-n}(U) = \{p\}$. Note that if f is invertible, then p is attracting fixed point for f if and only if it is repelling fixed point for f^{-1}, and vice versa.

Theorem 3.2.1. *Let T be a rotation map and let $a \in \mathbb{R}^2 \setminus \{0\}$ be such that $\alpha = \|a\| < 1$. Then, for $\cos\theta > 0$, $|\sin\theta| < \alpha$, $\overline{T}_a$ has two fixed points; one of them is attracting and other is repelling.*

Proof. Note that the fixed points of $\overline{T}_a$, for the rotation r, are $x_t = a(t-r)^{-1}$, where $t = \cos\theta \pm \sqrt{\alpha^2 - \sin^2\theta}$. Let $r_1 = \cos\theta > 0$ and $r_2 = \sin\theta$, $|r_2| < \alpha$.

As in Lemma 3.1.6, without loss of any generality we can choose $a = (0, \alpha)$. Let $x_{t_1} = (-r_2/\alpha, \sqrt{\alpha^2 - r_2^2}/\alpha)$ and let $x_{t_2} = (-r_2/\alpha, -\sqrt{\alpha^2 - r_2^2}/\alpha)$. Now these are the fixed points of $\overline{T}_a$.

Step I. Let $\psi : [0,1] \to \mathbb{S}^1$ be defined by $\psi(t) = e^{2\pi i(\theta + t)}$, where $e^{2\pi i\theta} = x_{t_1}$. Here, the restriction of ψ to $]0,1[$ is a homeomorphism and $\psi(0) = \psi(1) = x_{t_1}$. Since $x_{t_1} \neq x_{t_2}$, there exists $s_0 \in\,]0,1[$ such that $\psi(s_0) = x_{t_2}$. Let $\phi : [0,1] \to [0,1]$ be a map defined by $\phi(t) = (\psi^{-1} \circ \overline{T}_a \circ \psi)(t)$, for $0 < t < 1$, $\phi(0) = 0$ and $\phi(1) = 1$. Since $\overline{T}_a$ is an orientation preserving homeomorphism we have that ϕ is a homeomorphism and it is increasing. Observe that $\psi \circ \phi = \overline{T}_a \circ \psi$ and the set of fixed points of ϕ is $\{0, s_0, 1\}$. Note that $\phi(]0, s_0[) = \,]0, s_0[$ and $\phi(]s_0, 1[) = \,]s_0, 1[$.

Step II. Let $0 \leq r_2 < \alpha$. Let $s_1 \in\,]0, s_0[$ such that $\psi(s_1) = (-1, 0)$, where $(-1,0) \in \mathbb{S}^1$. Here, $\overline{T}_a((-1,0)) = (-r_1, \alpha - r_2)/\sqrt{1 + \alpha^2 - 2r_2\alpha}$. As $r_2 < \alpha$, $\phi(s_1) < s_1$. Since ϕ has no fixed point in $]0, s_0[$, $\phi(t) < t$ for all $t \in\,]0, s_0[$. As ϕ is an increasing function, for every $t \in\,]0, s_0[$. $\{\phi^n(t)\}$ is a decreasing sequence and hence $\phi^n(t) \to 0$.

As $r_2 \geq 0$ there exist $s_2, s_3 \in\,]s_0, 1[$ such that $\psi(s_2) = (1, 0)$, $\psi(s_3) =$

$\overline{T}_a(1,0) = (r_1, r_2+\alpha)/\sqrt{1+\alpha^2+2\alpha r_2}$. As $r_1 > 0$ and $r_2 \geq 0$, we have that $\phi(s_2) = s_3 > s_2$. Since there are no other fixed points between s_0 and 1, $\phi(t) > t$, for all $t \in \,]0, s_0[$. As ϕ is an increasing function, for every $t \in \,]s_0, 1[$. $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is an increasing sequence, and hence $\phi^n(t) \to 1$.

Note that for every t, if $0 < t < s_0$ (resp. $s_0 < t < 1$), $\phi(t) < t$ (resp. $\phi(t) > t$). This implies that for any neighbourhood $U_t = [0, t[\cup]1-t, 1]$ of 0 and 1 (resp. $V_t = \,]s_0 - t, s_0 + t[$ of s_0), where $t < \min\{s_0, 1-s_0\}$, $\phi(\overline{U_t}) \subset U_t$ and $\cap_{n\in\mathbb{N}}\phi^n(U_t) = \{0,1\}$ and $\overline{V_t} \subset \phi(V_t)$ and $\cap_{n\in\mathbb{N}}\phi^{-n}(V_t) = \{s_0\}$. Therefore, x_{t_1} is an attracting fixed point and x_{t_2} is a repelling fixed point for $\overline{T}_a$.

Step III. Let $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ be defined by $\varphi((x_1, x_2)) = (-x_1, x_2)$, and $\overline{T}'_a(x) = (a + r'x)/\|a + r'x\|$, where $r' = r_1 - ir_2$ and $x \in \mathbb{S}^1$. It is easy to see that $\varphi \circ \overline{T}_a = \overline{T}'_a \circ \varphi$, as φ is a linear map such that $\varphi(a) = \varphi(0, \alpha) = a$. Therefore, the dynamics of $\overline{T}_a$ and $\overline{T}'_a$ will be same, i.e. $\overline{T}^n_a(x) \to y$ if and only if $\overline{T}'^{\,n}_a(\varphi(x)) \to \varphi(y)$. Hence for $-\alpha < r_2 < 0$, x_{t_1} is an attracting and x_{t_2} is a repelling fixed point for $\overline{T}_a$. □

Corollary 3.2.2. *Let $T = \pm\mathrm{Id}$ and $a \in \mathbb{R}^2 \setminus \{0\}$ as in above Theorem 3.2.1, then*

(i) For $T(x) = x$, $\overline{a}$ and $-\overline{a}$ are the only fixed points of $\overline{T}_a$ and $\overline{T}^m_a(x) \to \overline{a}$, for all $x \in \mathbb{S}^1 \setminus \{-\overline{a}\}$, i.e., $\overline{a}$ is an attracting fixed point and $-\overline{a}$ is a repelling fixed point.

(ii) For $T(x) = -x$, $\overline{a}, -\overline{a}, x_0$ and $a - x_0$ are the only periodic points of $\overline{T}_a$ of period two where x_0 is such that $\|a - x_0\| = 1$ and $\{\overline{T}^{2m}_a(x)\}$ converges to either x_0 or $a - x_0$, for all $x \in \mathbb{S}^n \setminus \{\overline{a}, -\overline{a}\}$. In other words, $x_0, a - x_0$ are attracting fixed points and $\overline{a}, -\overline{a}$ are repelling fixed points for $\overline{T}^2_a$.

Proof. **Step 1.** Let $T(x) = x$. By Theorem 3.2.1, it is easy to see that $\overline{a}$ is an attracting fixed point and $-\overline{a}$ is a repelling fixed point for $\overline{T}_a$. That is, $\overline{T}_a^m(x) \to \overline{a}$, for all $x \in \mathbb{S}^1 \setminus \{-\overline{a}\}$.

Step 2. Let $T(x) = -x$. Then $\overline{T}_a(x) = (a - x)/\|a - x\|$. The set of periodic points of $\overline{T}_a$ of order two is $\{\overline{a}, -\overline{a}, x_0, a - x_0\}$, where x_0 is as in the hypothesis.

Consider, $n = 1$. Let $x \in \mathbb{S}^1 \setminus \{\overline{a}, -\overline{a}\}$ be fixed. Let ϑ (resp. ϑ') be the angle between a and x (resp. $-a$ and x), and ϑ_m (resp. ϑ'_m) be the angle between a and $\overline{T}_a^m(x)$ $\big($resp. $-a$ and $\overline{T}_a^m(x)\big)$, $m \in \mathbb{N}$. Then $\cos\vartheta = \langle a, x\rangle/\|a\|\|x\| = \langle a, x\rangle/\alpha$,

$\cos\vartheta_1 = \langle a, (a - x)/\|a - x\|\rangle/\alpha = (\alpha - \cos\vartheta)/c_1$, where $c_1 = \|a - x\|$, and

$\cos\vartheta_2 = \langle a, \overline{T}_a((a - x)/c_1)\rangle/\alpha = ((c_1 - 1)\alpha + \cos\vartheta)/c_2$, where $c_2 = \|(c_1 - 1)a + x\|$.

As $c_2 \geq 1 - |c_1 - 1|\alpha$, $\cos\vartheta_2 - |c_1 - 1|\alpha\cos\vartheta_2 \leq (c_1 - 1)\alpha + \cos\vartheta$.

If $c_1 < 1$, then we get that $\cos\vartheta_2 < \cos\vartheta$. That is,

$$\text{if } \|a - x\| < 1 \text{ then } \|\overline{a} - \overline{T}_a^2(x)\| > \|\overline{a} - x\|. \tag{1}$$

Similarly, we can show that, if $c_1 > 1, \cos{\vartheta_2}' < \cos\vartheta'$, where $\cos\vartheta' = \langle -a, x\rangle/\alpha$ and $\cos{\vartheta_2}' = \langle -a, \overline{T}_a^2(x)\rangle/\alpha$. That is,

$$\text{if } \|a - x\| > 1 \text{ then } \|\overline{a} + \overline{T}_a^2(x)\| > \|\overline{a} + x\|. \tag{2}$$

Let $\psi : [0, 1] \to \mathbb{S}^1$ defined by $\psi(t) = e^{2\pi i(r+t)}$, where $e^{2\pi ir} = \overline{a}$. Here, the restriction of ψ to $]0, 1[$ is a homeomorphism and $\psi(0) = \psi(1) = \overline{a}$. Then there exist $t_0, t_1 \in]0, 1[$

such that $\psi(t_0) = x_0$ and $\psi(t_1) = a - x_0$. Interchanging x_0 and $a - x_0$, if necessary, we may assume that $t_0 < t_1$. Note that as $\|a - x_0\| = 1$ and $\psi(1/2) = -\overline{a}$, we get that $t_0 < 1/2 < t_1$. Let $\phi : [0,1] \to [0,1]$ be the homeomorphism defined by $\phi(t) = (\psi^{-1} \circ \overline{T}_a^2 \circ \psi)(t)$, for $0 < t < 1$ and $\phi(0) = 0, \phi(1) = 1$. Observe that $\psi \circ \phi = \overline{T}_a^2 \circ \psi$ and the set of fixed points of ϕ is $\{0, t_0, t_1, 1\}$. From (1), it follows that if $t \in \,]0, t_0[$ (resp. $t \in \,]t_1, 1[$), then $\phi(t) > t$ (resp. $\phi(t) < t$). Since ϕ is monotone, and in particular, increasing, it follows that $\{\phi^m(t)\}_{m \in \mathbb{N}}$ is an increasing sequence (resp. a decreasing sequence) which is bounded above (resp. below) by t_0 (resp. t_1), if $t \in \,]0, t_0[$ (resp. $t \in \,]t_1, 1[$), and hence it converges to a fixed point in $]0, t_0]$ (resp. $[t_1, 1[$). That is, $\phi^m(t) \to t_0$ for $t \in \,]0, t_0[$ and $\phi^m(t) \to t_1$ for $t \in \,]t_1, 1[$. Similarly using (2) we get that, if $t \in \,]t_0, 1/2[$ (resp. $t \in \,]1/2, t_1[$), $\phi(t) < t$ (resp. $\phi(t) > t$). Therefore, $\phi^m(t) \to t_0$ for $t \in \,]t_0, 1/2[$ and $\phi^m(t) \to t_1$ for $t \in \,]1/2, t_1[$. As ψ is continuous and $\psi \circ \phi = \overline{T}_a^2 \circ \psi$, the following holds: If $x \in \psi(]0, 1/2[)$ (resp. $x \in \psi(]1/2, 1[)$), then $\overline{T}_a^{2m}(x) \to x_0, \overline{T}_a^{2m+1}(x) \to a - x_0$ (resp. $\overline{T}_a^{2m}(x) \to a - x_0, \overline{T}_a^{2m+1}(x) \to x_0$). This shows that $x_0, a - x_0$ are attracting fixed points for $\overline{T}_a^2$ and $\overline{a}, -\overline{a}$ are repelling fixed points for $\overline{T}_a^2$. Hence (ii) holds. □

Chapter 4

Dynamics of linear and 'affine' actions on p-adic unit spheres

In this chapter we consider the n-dimensional p-adic vector space $\mathbb{Q}_p^n$ and study the dynamics of linear and 'affine' actions on p-adic unit sphere $\mathcal{S}_n$. This Chapter includes p-adic analogues of results about linear and 'affine' actions of semigroups (of $GL(n+1,\mathbb{R})$) on unit sphere $\mathbb{S}^n$ discussed in Chapter 2 and Chapter 3. In the first section we consider a semigroup $\mathfrak{S}$ of $GL(n,\mathbb{Q}_p)$ and study the distality of the $\mathfrak{S}$-action on $\mathcal{S}_n$. In the second section, for $T \in GL(n,\mathbb{Q}_p)$ and a certain set of nonzero a, we define 'affine' maps $\overline{T}_a$ on p-adic unit sphere $\mathcal{S}_n$ and study the dynamics of 'affine' homeomorphisms $\overline{T}_a$ on $\mathcal{S}_n$.

4.1 Distality of the semigroup actions on $\mathcal{S}_n$

Let $\mathbb{Q}_p$ be the p-adic field, and $|\cdot|_p$ denote the p-adic absolute value on $\mathbb{Q}_p$. For $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, $n \in \mathbb{N}$, let $\|x\|_p = \max_{1\leq i\leq n} |x|_p$, which defines a p-adic norm on the vector space $\mathbb{Q}_p^n$. Let $\mathcal{S}_n = \{x \in \mathbb{Q}_p^n \mid \|x\|_p = 1\}$ be the p-adic unit sphere. For disquisition on p-adic analysis we refer Koblitz [22].

For $T \in GL(n, \mathbb{Q}_p)$, let $\|T\|_p = \sup\{\|T(x)\|_p \mid x \in \mathbb{Q}_p, \|x\|_p = 1\}$. Observe that the norm of an element or a matrix, defined this way, is of the from p^m for some $m \in \mathbb{Z}$. We call $T \in GL(n, \mathbb{Q}_p)$ an *isometry* if it preserves the norm, i.e. if T keeps $\mathcal{S}_n$ invariant. Note that T is an isometry if and only if $\|T\|_p = 1 = \|T^{-1}\|_p$. For $x, y \in \mathbb{Q}_p^n$, $\|x + y\|_p \leq \max\{\|x\|_p, \|y\|_p\}$; the equality holds if $\|x\|_p \neq \|y\|_p$. We will use this fact extensively. Recall that for a $T \in GL(n, \mathbb{Q}_p)$, $\overline{T} : \mathcal{S}_n \to \mathcal{S}_n$ is defined as $\overline{T}(x) = \|T(x)\|_p(T(x))$, for all $x \in \mathcal{S}_n$. In this section we consider the group action of $GL(n, \mathbb{Q}_p)$ on $\mathcal{S}_n$. For semigroups of $GL(n, \mathbb{Q}_p)$, we prove a result analogous to Theorem 2.2.1 (see Theorem 4.1.4). Recall that $T \in GL(n, \mathbb{Q}_p)$ is said to be distal if $\{T^m\}_{m\in\mathbb{Z}}$ acts distally on $\mathbb{Q}_p^n$.

For an invertible linear map T on a p-adic vector space $V = \mathbb{Q}_p^n$, let $C(T) = \{v \in V \mid T^m(v) \to 0 \text{ as } m \to \infty\}$. Here, $V = C(T) \oplus M(T) \oplus C(T^{-1})$, where $M(T) = \{v \in V \mid \{T^m(v)\}_{m\in\mathbb{Z}} \text{ is relatively compact}\}$. Note that T acts distally on V if and only if $C(T)$ and $C(T^{-1})$ are trivial (see Proposition 2.1 in Jaworski-Raja [20], which is based on results of Baumgartener-Willis [11]). We refer the reader to Wang [33] for more details on the structure of p-adic contraction spaces. We will use the notion of contraction spaces below.

Lemma 4.1.1. *Let $T \in GL(n, \mathbb{Q}_p)$. The following are equivalent:*

(1) *T is distal.*

(2) *The closure of the group generated by T in $GL(n, \mathbb{Q}_p)$ is compact.*

(3) *T^m is an isometry for some $m \in \mathbb{N}$.*

Proof. (3) $\Rightarrow$ (2) is obvious and (2) $\Rightarrow$ (1) follows as compact groups act distally. Now suppose T is distal, i.e. $\{T^m\}_{m\in\mathbb{Z}}$ acts distally on $\mathbb{Q}_p^n$. Then the contraction spaces $C(T)$ and $C(T^{-1})$ are trivial. By Lemma 3.4 of [33], we get that $\mathbb{Q}_p^n = M(T) = \{x \in \mathbb{Q}_p^n \mid \{T^m(x)\}_{m\in\mathbb{Z}}$ is relatively compact$\}$, (cf. [33]). By Proposition 1.3 of [33], $\cup_{m\in\mathbb{Z}} T^m(\mathcal{S}_n)$ is relatively compact, i.e. $\{\|T^m\|_p\}_{m\in\mathbb{Z}}$ is bounded and hence $\{T^m \mid m \in \mathbb{Z}\}$ is relatively compact in $GL(n, \mathbb{Q}_p)$. This proves (1) $\Rightarrow$ (2). Now suppose T is contained in a compact group. Then $T^{\pm m_k} \to \mathrm{Id}$, for some $\{m_k\} \subset \mathbb{N}$. Therefore, $\|T^{\pm m_k}\|_p \to 1$ and as $\{\|T^m\|_p \mid m \in \mathbb{Z}\} \subset \{p^l \mid l \in \mathbb{Z}\}$, we get that for all large k, $\|T^{\pm m_k}\|_p = 1$ and T^{m_k} is an isometry. Therefore, (2) $\Rightarrow$ (3). $\square$

We state a useful result which is well-known and can be proven easily.

Lemma 4.1.2. *Let X be a locally compact (Hausdorff) topological space and let* Homeo(X) *be the topological group homeomorphisms of X endowed with the compact open topology. Let $A, B \in$* Homeo(X) *be such that $AB = BA$ and B generates a relatively compact group in* Homeo(X). *Then A is distal if and only if AB is distal.*

We now consider a canonical group action of $GL(n, \mathbb{Q}_p)$ on $\mathcal{S}_n$: For $T \in GL(n, \mathbb{Q}_p)$ and $x \in \mathcal{S}_n$, $\overline{T}(x) = \|T(x)\|_p T(x)$. Observe that $\mathcal{S}_1 = \mathbb{Z}_p^* = \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$ and $GL(1, \mathbb{Q}_p) = \mathbb{Q}_p \setminus \{0\}$ acts distally on $\mathcal{S}_1$ as $\overline{T} = \|T\|_p T \in \mathcal{S}_1$ for every $T \in GL(1, \mathbb{Q}_p)$. The following will be useful in proving the main result of this section.

Proposition 4.1.3. *Let $T \in GL(n, \mathbb{Q}_p)$. If bT is distal for some $b \in \mathbb{Q}_p$, then $\overline{T}$ is distal. Conversely, if $\overline{T}$ is distal, then for some $m \in \mathbb{N}$ and $l \in \mathbb{Z}$, $p^l T^m$ is distal. If $|\det T|_p = 1$ and $\overline{T}$ is distal, then T is distal.*

Proof. Observe that bT is distal if and only if $|b|_p^{-1}T$ is so. As $\overline{T} = \overline{p^m T}$ for any $m \in \mathbb{Z}$, we may replace T by $|b|_p^{-1}T$ and assume that T is distal. By Lemma 4.1.1, T generates a relatively compact group, and hence $\overline{T}$ is distal. Conversely, suppose $\overline{T}$ is distal. By 3.3 of [33], there exists $m \in \mathbb{N}$ such that $T^m = AUC$, where C is a diagonal matrix, U is unipotent, A is semisimple, A, U and C commute with each other and A as well as U generate a relatively compact group. Now by Lemma 4.1.2, we have that $\overline{C}$ is distal. Here, $C = DD' = D'D$ for some diagonal matrices D and D' such that the diagonal entries of D (resp. D') are of the form p^{l_k}, $l_k \in \mathbb{Z}$, $k = 1, \ldots n$ (resp. in $\mathbb{Z}_p^*$). Since D' also generates a relatively compact group and it commutes with D, by Lemma 4.1.2, $\overline{D}$ is distal. It is enough to show that $D = p^l \mathrm{Id}$, as in this case, D would be central in $GL(n, \mathbb{Q}_p)$ and this would imply that AU and D' commute, and hence, $p^{-l}T^m = AUD'$ would generate a relatively compact group which in turn would imply that it is distal. If possible, suppose p^l and p^{l_1} are two entries in D such that $l < l_1$. As $\overline{D} = \overline{p^{-l}D}$, we have that $D_1 = p^{-l}D$ is distal, 1 is an eigenvalue of D_1 and D_1 has another eigenvalue $p^{l_1 - l}$ which has p-adic absolute value less than 1. Then the contraction space of D_1, $C(D_1) \neq \{0\}$ as we can take a nonzero $y \in \mathbb{Q}_p^n$ satisfying $D_1(y) = p^k y$ for $k = l_1 - l \in \mathbb{N}$; and it follows that $y \in C(D_1)$. Let $x \in \mathcal{S}_n$ be such that $D_1(x) = x$ and let y be as above such that $0 < \|y\|_p < 1$. Then $\overline{D_1}(x) = x$ and $x + y \in \mathcal{S}_n$. For $D_1^i(x+y) = (x + p^{ki}y) \to x \in \mathcal{S}_n$ as $i \to \infty$. Therefore, $\overline{D_1}^i(x+y) \to x$ and it leads to a contradiction as $\overline{D_1}$ is distal. Therefore, $D = p^{-l}\mathrm{Id}$ and $p^{-l}T^m$ is distal.

Suppose $|\det T|_p = 1$. Then $|\det(T^m)|_p = |\det T|_p^m = 1$. As $\overline{T}$ is distal,

$T^m = p^l S$ for some $l \in \mathbb{Z}$ where S generates a relatively compact group. Then $|\det S|_p = 1$, and hence $l = 0$ and $T^m = S$. $\square$

The following Theorem characterises distal actions of semigroups on $\mathcal{S}_n$.

Theorem 4.1.4. *Let $\mathfrak{S} \subset SL(n, \mathbb{Q}_p)$ be a semigroup. Then the following are equivalent:*

1. *$\mathfrak{S}$ acts distally on $\mathcal{S}_n$.*
2. *The group generated by $\mathfrak{S}$ acts distally on $\mathcal{S}_n$.*
3. *The closure of $\mathfrak{S}$ is a compact group.*

Proof. Suppose (1) holds. Let $\mathfrak{S} \subset SL(n, \mathbb{Q}_p)$. As the closure $\overline{\mathfrak{S}}$ of $\mathfrak{S}$ is a semigroup in $SL(n, \mathbb{Q}_p)$ and it also acts distally on $\mathcal{S}_n$, we may assume that $\mathfrak{S}$ is closed. By Proposition 4.1.3 and Lemma 4.1.1, each element in $\mathfrak{S}$ generates a relatively compact group (in $\mathfrak{S}$). In particular, each element of $\mathfrak{S}$ has an inverse in $\mathfrak{S}$ and $\mathfrak{S}$ is a group. Now by Lemma 3.3 of [17], $\mathfrak{S}$ is contained in a compact extension of a unipotent subgroup $\mathcal{U}$ in $GL(n, \mathbb{Q}_p)$ which is normalised by $\mathfrak{S}$, i.e. $\mathfrak{S} \subset K \ltimes \mathcal{U}$, where K is a compact group which normalises $\mathcal{U}$.

By Kolchin's Theorem, there exists a flag $\{0\} = V_0 \subset \cdots \subset V_k = \mathbb{Q}_p^n$ of maximal $\mathcal{U}$-invariant subspaces such that $\mathcal{U}$ acts trivially on V_j/V_{j-1}, $j = 1, \ldots, k$. Note that each V_j is maximal in the sense that for any subspace W containing V_j such that $W \neq V_j$, $\mathcal{U}$ does not act trivially on W/V_{j-1}. It is easy to see that each V_j is $\mathfrak{S}$-invariant as $\mathfrak{S}$-normalises $\mathcal{U}$. If possible, suppose $\mathfrak{S}$ is not compact. Then there exists a sequence $\{T_i\} \subset \mathfrak{S}$ such that $\{T_i\}$ is unbounded. Then $T_i = K_i U_i$,

$i \in \mathbb{N}$, where $K_i \in K$ and $U_i \in \mathcal{U}$ such that $\{K_i\}$ is relatively compact and $\{U_i\}$ is unbounded. Note that for each j, as $\mathfrak{S}$ and $\mathcal{U}$ keep V_j invariant, $K_i(V_j) = V_j$ for all i. Passing to a subsequence if necessary, we get that there exists $w \in \mathcal{S}_n$ such that $\|U_i(w)\|_p \to \infty$, $\overline{T}_i(w) \to w' \in \mathcal{S}_n$ and also that $K_i \to K_0$. For every $v \in V_1$, $T_i(v) = K_i(v) \in V_1$, and hence $\{\|T_i(v)\|_p\}$ is bounded. Let $v \in V_1 \setminus \{0\}$ be such that $\|v\|_p < 1$. Then $v + w \in \mathcal{S}_n$. As $\{\|K_iU_i(v+w)\|_p\}$ is unbounded and $K_iU_i(v) = K_i(v) \to K_0 v$, we get that $\overline{T}_i(v+w) = \overline{K_iU_i}(v+w) \to w'$. This contradicts (1). Therefore, $\overline{\mathfrak{S}}$ is compact, i.e. (1) $\Rightarrow$ (3).

Suppose $\overline{\mathfrak{S}}$ is a compact group. Then it contains G, where G is the group generated by $\mathfrak{S}$. Since G is relatively compact, G acts distally on $\mathcal{S}_n$. Hence (3) $\Rightarrow$ (2). It is obvious that (2) $\Rightarrow$ (1). $\square$

Note that the above Theorem 4.1.4 is valid for a semigroup $\mathfrak{S} \subset GL(n, \mathbb{Q}_p)$ satisfying the condition that $|\det(T)|_p = 1$ for all $T \in \mathfrak{S}$; as if T is distal then by above Proposition 4.1.3 the above condition implies that each T generates a compact group and hence the rest of proof follows as it is. Consider $\mathcal{D} = \{b\,\mathrm{Id} \mid \mathrm{b} \in \mathbb{Q}_\mathrm{p}\}$, the centre of $GL(n, \mathbb{Q}_p)$. It acts trivially on $\mathcal{S}_n$ and the group action of $GL(n, \mathbb{Q}_p)$ on $\mathcal{S}_n$ factors through $\mathcal{D}$. One can get an exact p-adic analogue of Theorem 4.1.4 (1-2) for semigroups of $GL(n, \mathbb{Q}_p)$ using the techniques of algebraic groups (cf. [31]).

In the real case, Corollary 2.2.5 showed that $\mathfrak{S} \subset SL(n+1, \mathbb{R})$ acts distally on the (real) unit sphere $\mathbb{S}^n$ if (and only if) every cyclic subsemigroup of $\mathfrak{S}$ acts distally. An analogous statement does not hold in the p-adic case, as there exists a class of closed non-compact subgroups of $SL(n, \mathbb{Q}_p)$ whose every cyclic subgroup is relatively compact but it does not act distally on $\mathcal{S}_n$ as it is not compact; e.g. the group of strictly upper triangular matrices in $SL(n, \mathbb{Q}_p)$, $n \geq 2$.

4.2 Distality of 'affine' maps on p-adic unit spheres $\mathcal{S}_n$

In this section, we discuss the 'affine' actions on the p-adic unit sphere $\mathcal{S}_n$. Consider the affine action on $\mathbb{Q}_p^n$, $T_a(x) = a + T(x)$, where $T \in GL(n, \mathbb{Q}_p)$, and $a \in \mathbb{Q}_p^n$. We first consider the corresponding 'affine' map $\overline{T}_a$ on $\mathcal{S}_n$ which is defined for any nonzero a satisfying $\|T^{-1}(a)\| \neq 1$ as follows: $\overline{T}_a(x) = \|T_a(x)\|_p(T_a(x))$, $x \in \mathcal{S}_n$. (For $a = 0$, $\overline{T}_a = \overline{T}$, which is studied in the previous section of this chapter). Observe that $T_a(x) = 0$ for some $x \in \mathcal{S}_n$ if and only if $T^{-1}(a)$ has norm 1. Therefore, $\overline{T}_a$ is well defined if $\|T^{-1}(a)\|_p \neq 1$. The map $\overline{T}_a$ is a homeomorphism for any nonzero a satisfying $\|T^{-1}(a)\| < 1$ (see Lemma 4.2.1 below). In this section, we study the distality of such homeomorphisms $\overline{T}_a$.

Lemma 4.2.1. *Let $T \in GL(n, \mathbb{Q}_p)$ and let $a \in \mathbb{Q}_p^n \setminus \{0\}$ be such that $\|T^{-1}(a)\|_p \neq 1$. Then the map $\overline{T}_a$ on $\mathcal{S}_n$ is continuous and injective. $\overline{T}_a$ is a homeomorphism if and only if $\|T^{-1}(a)\|_p < 1$.*

Proof. Suppose $\|T^{-1}(a)\|_p \neq 1$. From the definition of $\overline{T}_a$, it is obvious that it is continuous. Suppose $x, y \in \mathcal{S}_n$ such that $\overline{T}_a(x) = \overline{T}_a(y)$. Then

$$\|a + T(x)\|_p \,(a + T(x)) = \|a + T(y)\|_p \,(a + T(y))$$

or $(\beta - 1)T^{-1}(a) = y - \beta x$, where $\beta = \|a+T(x)\|_p / \|a+T(y)\|_p = p^m$ for some $m \in \mathbb{Z}$. If possible suppose $\beta \neq 1$. Interchanging y and x if necessary, we may assume that $\beta > 1$ or equivalently, that $m \in \mathbb{N}$. This implies that $\|\beta x\|_p = |\beta|_p = p^{-m} < 1$, and we get

$$\|T^{-1}(a)\|_p = |\beta - 1|_p \|T^{-1}(a)\|_p = \|y - \beta x\|_p = 1,$$

a contradiction. Hence, $\beta = 1$ and $x = y$. Therefore, $\overline{T}_a$ is injective.

Now suppose $\|T^{-1}(a)\|_p < 1$. It is enough to show that $\overline{T}_a$ is surjective, as any continuous bijection on a compact Hausdorff space is a homeomorphism.

Let $y \in \mathcal{S}_n$. Let $z = T^{-1}(y)$ and let $x = \|z\|_p z - T^{-1}(a)$. Since the norm of $\|z\|_p z$ is 1 and $\|T^{-1}(a)\|_p < 1$, we have that $\|x\|_p = 1$. Moreover, as $\|y\|_p = 1$, we have that $\|z\|_p^{-1} = \|a + T(x)\|_p$. Therefore, $\overline{T}_a(x) = y$. Hence $\overline{T}_a$ is surjective.

Conversely, Suppose $\overline{T}_a$ is surjective. Then there exists $x \in \mathcal{S}_n$ such that $\overline{T}_a(x) = \|a\|_p a$. We get that $x = (p^m - 1)T^{-1}(a)$, where $p^m = \|a + T(x)\|_p^{-1}\|a\|_p$ for some $m \in \mathbb{Z}$; here $m \neq 0$ since $x \neq 0$. Now $1 = \|x\|_p = |p^m - 1|_p \|T^{-1}(a)\|_p \geq \|T^{-1}(a)\|_p$ since $|p^m - 1|_p \geq 1$ for every $m \in \mathbb{Z} \setminus \{0\}$. As $\|T^{-1}(a)\|_p \neq 1$, we have that $\|T^{-1}(a)\|_p < 1$. $\square$

In Chapter 3, we have studied 'affine' maps $\overline{T}_a$ on the real unit sphere $\mathbb{S}^n$. The following result shows that in the p-adic case, $\overline{T}_a$ is distal for every nonzero a in a certain neighbourhood of 0 in $\mathbb{Q}_p^n$ if and only if $\overline{T}$ is so. This illustrates that the behaviour of such maps in the p-adic case is very different from that in the real case.

Theorem 4.2.2. *Suppose $T \in GL(n, \mathbb{Q}_p)$. Let $\overline{T}_a : \mathcal{S}_n \to \mathcal{S}_n$ be defined as $\overline{T}_a(x) = \|a + T(x)\|_p (a + T(x))$, $x \in \mathcal{S}_n$. There exists an open compact group V such that for all $a \in V \setminus \{0\}$ we have $\|T^{-1}(a)\|_p < 1$ and the following hold:*

(I) *If $\overline{T}$ is distal, then $\overline{T}_a$ is distal for all nonzero $a \in V$.*

(II) *If $\overline{T}$ is not distal, then for every neighbourhood U of 0 contained in V, there exists a nonzero $a \in U$ such that $\overline{T}_a$ is not distal.*

Proof. By 3.3 of [33], we get that there exist D and S which commute with T and $m \in \mathbb{N}$ such that $T^m = SD = DS$, where D is a diagonal matrix with the diagonal entries in $\{p^i \mid i \in \mathbb{Z}\}$ and S generates a relatively compact group. Therefore, S^k is an isometry for some $k \in \mathbb{N}$. Replacing m by km, we may assume that S itself is an isometry. Let $c_0 = \min\{(1/\|T^{-j}\|_p) \mid 1 \leq j \leq m-1\}$ and $c_1 = \max\{\|T^j\|_p \mid 1 \leq j \leq m-1\}$. As $\mathcal{S}_n$ is compact, $0 < c_0 \leq c_1 < \infty$. Also, $c_0 \leq \|T^j(x)\|_p \leq c_1$ for all $x \in \mathcal{S}_n$ and $1 \leq j \leq m-1$. Since $\|T^j\|_p \in \{p^i \mid i \in \mathbb{Z}\}$, we get that $\{\|T^j(x)\|_p \mid x \in \mathcal{S}_n, 1 \leq j \leq m-1\}$ is finite.

Let V be an open compact S-invariant subgroup in $\mathbb{Q}_p^n$ such that $V \cup c_0 V \cup c_0^2 V \cup c_0^2 c_1^{-2} V \subset W = \{w \in \mathbb{Q}_p^n \mid \|w\|_p < 1\}$. Therefore, $\|v\|_p < \min\{1, c_0, c_0^2, c_0^2 c_1^{-2}\} \leq 1$ for all $v \in V$ and $c_0 c_1^{-1} V \subset c_0^2 c_1^{-2} V \subset W$. Moreover, $\|T^{-1}(v)\|_p < c_0^{-1} c_0 = 1$ for every $v \in V$.

Let p^l be the smallest nonzero entry in the diagonal matrix D and let $H = \{x \in \mathbb{Q}_p^n \mid D(x) = p^l x\}$. This is a nontrivial closed subspace of $\mathbb{Q}_p^n$. As S and T commute with D, they keep H invariant and, as S is an isometry, $\|T^m(x)\|_p = p^{-l}\|x\|_p$ for all $x \in H$.

Let $a \in V \setminus \{0\}$. Take any $x \in \mathcal{S}_n$. Since $\|a\|_p < c_0$ and $\|T(x)\|_p \geq c_0$, we have $\|T_a(x)\|_p = \|a + T(x)\|_p = \|(T(x)\|_p$ and

$$\overline{T}_a(x) = \|T_a(x)\|_p T_a(x) = \|T(x)\|_p (a + T(x)).$$

Let $\alpha_1(x) = \|T_a(x)\|_p = \|T(x)\|_p = \beta_{1,x}$. Let $\alpha_j(x) = \|T_a(\overline{T}_a^{j-1}(x))\|_p = \|a + T(\overline{T}_a^{j-1}(x))\|_p$, $j \in \mathbb{N}$, and let $\beta_{j,x} = \alpha_1(x) \cdots \alpha_j(x)$, for all $j \geq 2$. Take $\beta_{0,x} = 1$ and $\phi^0 = \mathrm{Id}$ for any map ϕ. From above, we have that $\alpha_j(x) = \|T(\overline{T}_a^{j-1}(x))\|_p$ for all

$j \in \mathbb{N}$. It is easy to show by induction that for every $j \in \mathbb{N}$,

$$\overline{T}_a^j(x) = \beta_{j,x} T^j(x) + \beta_{j,x} \sum_{i=1}^{j} \beta_{j-i,x}^{-1} T^{i-1}(a). \tag{4.1}$$

Observe that as $a \in V$, $\|T^k(a)\|_p \le c_1 \|a\|_p < c_1(c_0 c_1^{-1}) = c_0$ and for any $x \in \mathcal{S}_n$, $\|T^k(x)\|_p \ge c_0$, $1 \le k \le m-1$. Therefore, for $j \in \mathbb{N}$ and $1 \le k \le m-2$,

$$\begin{aligned} \|T^k(\overline{T}_a^j(x))\|_p &= [\alpha_j(x)]^{-1} \|T^k(a) + T^{k+1}(\overline{T}_a^{j-1}(x))\|_p \\ &= [\alpha_j(x)]^{-1} \|T^{k+1}(\overline{T}_a^{j-1}(x))\|_p. \end{aligned}$$

Applying the above equation successively, we get that for $1 \le j \le m-1$, $\alpha_j(x) = \|T(\overline{T}_a^{j-1}(x))\|_p = [\alpha_{j-1}(x) \cdots \alpha_1(x)]^{-1} \|T^j(x)\|_p$ i.e. $\beta_{j,x} = \|T^j(x)\|_p$. Hence, $c_0 \le \beta_{j,x} \le c_1$ for all $x \in \mathcal{S}_n$ and $1 \le j \le m-1$. Moreover, applying the same equation again successively, we get for $j \ge m$ that

$$\alpha_j(x) = [\alpha_{j-1}(x) \cdots \alpha_{j-m+1}(x)]^{-1} \|T^{m-1}(a) + T^m(\overline{T}_a^{j-m}(x))\|_p. \tag{4.2}$$

Now we take $a \in V \cap H \setminus \{0\}$. Let $x \in \mathcal{S}_n \cap H$. Then $\|T^{-1}(a)\|_p < c_0^{-1} c_0 = 1$ and hence, $\|T^{-1}(a) + \overline{T}^{j-m}(x)\|_p = 1$. This implies that

$$\|T^{m-1}(a) + T^m(\overline{T}^{j-m}(x))\|_p = \|T^m(T^{-1}(a) + \overline{T}^{j-m}(x))\|_p = p^{-l}. \tag{4.3}$$

Using Eqs. (2) and (3), we get $\alpha_j(x) = [\alpha_{j-1}(x) \cdots \alpha_{j-m+1}(x)]^{-1} p^{-l}$, and hence $\beta_{j,x} = p^{-l} \beta_{j-m,x}$ for all $j \ge m$. In particular, $\beta_{m,x} = p^{-l} = \|T^m(x)\|_p$. This implies that $\beta_{km+j,x} = p^{-kl} \beta_{j,x} = p^{-kl} \|T^j(x)\|_p$, $k, j \in \mathbb{N}$. Therefore, $\beta_{j,x} = \|T^j(x)\|_p$, $j \in \mathbb{N}$. Moreover, for all $j, k \in \mathbb{Z}$ and $x \in H$, $T^{km+j}(x) = p^{kl} S^k T^j(x) = p^{kl} T^j S^k(x)$ and, $\|T^{km+j}(x)\|_p = p^{-kl} \|T^j(x)\|_p$ as S is an isometry. In particular, $\beta_{km+j,x} = p^{-kl} \|T^j(x)\|_p$ for all $k, j \in \mathbb{Z}$ such that $km + j \ge 0$. Using the above facts together with Eq. (1), we get for $k \in \mathbb{N}$,

$$\begin{aligned}
\overline{T}_a^{km}(x) &= \beta_{km,x} T^{km}(x) + \beta_{km,x} \sum_{j=1}^{km} \beta_{km-j,x}^{-1} T^{j-1}(a) \\
&= S^k(x) + \sum_{j=1}^{km} \|T^{-j}(x)\|_p^{-1} T^{j-1}(a) \\
&= S^k(x) + \sum_{i=1}^{k} \sum_{j=1}^{m} \|T^{-j}(x)\|_p^{-1} T^{j-1}(S^{i-1}(a)) \\
&= S^k(x) + \sum_{j=1}^{m} \gamma_{j,x}^{-1}\, T^{j-1}(a_k),
\end{aligned}$$

where $a_k = \sum_{i=1}^{k} S^{i-1}(a) \in V \cap H$, $k \in \mathbb{N}$, $\gamma_{j,x} = \|T^{-j}(x)\|_p = p^l \beta_{m-j,x}$ and $c_1^{-1} \leq \gamma_{j,x} \leq c_0^{-1}$, $1 \leq j \leq m-1$, and $\gamma_{m,x} = p^l$. From above, we get that for any $k \in \mathbb{N}$ and $x, y \in \mathcal{S}_n \cap H$,

$$\overline{T}_a^{km}(x) - \overline{T}_a^{km}(y) = S^k(x-y) + \sum_{j=1}^{m-1} [\gamma_{j,x}^{-1} - \gamma_{j,y}^{-1}]\, T^{j-1}(a_k). \tag{4.4}$$

Let $x, y \in \mathcal{S}_n \cap H$ such that $\|x-y\|_p < c_0 c_1^{-1}$. As T is linear, $T^j(x) = T^j(y) + T^j(x-y)$, $j \in \mathbb{N}$. For $1 \leq j \leq m-1$, as $\|T^j(x-y)\|_p \leq c_1 \|x-y\|_p < c_0$, and $\|T^j(y)\|_p \geq c_0$, we get that $\beta_{j,x} = \|T^j(x)\|_p = \|T^j(y)\|_p = \beta_{j,y}$, and hence $\gamma_{j,x} = \gamma_{j,y}$. Therefore,

$$\|\overline{T}_a^{km}(x) - \overline{T}_a^{km}(y)\|_p = \|S^k(x) - S^k(y)\|_p = \|x-y\|_p, \ k \in \mathbb{N}.$$

Now suppose $\|x-y\|_p \geq c_0 c_1^{-1}$. Observe that $|\gamma_{j,x}^{-1} - \gamma_{j,y}^{-1}|_p \leq c_0^{-1}$, $\|T^j(a_k)\|_p \leq c_1 \|a_k\|_p$, $1 \leq j \leq m-1$ and $a_k \in V$, $\|a_k\|_p < c_0^2 c_1^{-2}$. Now Eq. (4) implies that $\overline{T}_a^{km}(x) - \overline{T}_a^{km}(y) \in S^k(x-y) + c_0^{-1} c_1 W$. Since $\|S^k(x-y)\|_p = \|x-y\|_p \geq c_0 c_1^{-1}$, we get that $\|\overline{T}_a^{km}(x) - \overline{T}_a^{km}(y)\|_p = \|x-y\|_p$. This shows that $\overline{T}_a^{m}|_H$ preserves the distance and its action on H is distal, where $a \in V \cap H$.

If $\overline{T}$ is distal, then so is $\overline{T}^m$, and hence its image in $GL(n, \mathbb{Q}_p)/\mathcal{D}$ generates a relatively compact group. This implies that $D = p^l\mathrm{Id}$, $H = \mathbb{Q}_p^n$ and $V \cap H = V$. Therefore, (I) holds.

Now suppose $\overline{T}$ is not distal. Then $\overline{T}^m$ is not distal and hence $D \neq p^l\mathrm{Id}$. Let $l_1 > l$ and $H_1 = \{x \in \mathbb{Q}_p^n \mid D(x) = p^{l_1}x\}$. Then H_1 is a vector subspace and it is invariant under D, S and T. For $a \in V \cap H \setminus \{0\}$ as above, we show that the action of $\overline{T}_a$ on $\mathcal{S}_n \cap (H \oplus H_1)$ is not distal. This would imply that (II) holds.

Take $y = x + z \in \mathcal{S}_n$, where $x \in \mathcal{S}_n \cap H$ and $z \in H_1$ such that $\|T^j(z)\|_p < \|T^j(x)\|_p$, $j \in \mathbb{N}$. It is possible to choose such a z; we can take $z \in H_1$ with the property that $\|T^j(z)\|_p < \|T^j(x)\|_p$ for all $0 \leq j \leq m-1$, then as S is an isometry, $\|T^{km+j}(z)\|_p = p^{-kl_1}\|T^j(z)\|_p < p^{-kl}\|T^j(x)\|_p = \|T^{km+j}(x)\|_p$, $k \in \mathbb{N}$. Now $\|T^j(y)\|_p = \|T^j(x)\|_p = \beta_{j,x}$ for all $j \in \mathbb{N}$. Here,

$$\overline{T}^{km}(y) - \overline{T}^{km}(x) = p^{-kl}[S^k(p^{kl}x + p^{kl_1}z)] - S^k(x) = p^{k(l_1-l)}S^k(z) \to 0$$

as $k \to \infty$, since S is an isometry and $l_1 > l$. We now show for all $k \in \mathbb{N}$ that $\overline{T}_a^{km}(y) - \overline{T}_a^{km}(x) = \overline{T}^{km}(y) - \overline{T}^{km}(x)$. (From above, the latter is equal to $\beta_{km,x}T^{km}(z)$.) This in turn would imply that $\overline{T}_a$ is not distal.

From Eq. (1), it is enough to show for all $j \in \mathbb{N} \cup \{0\}$ that $\beta_{j,y} = \beta_{j,x}$, or equivalently, $\beta_{j,y} = \|T^j(y)\|_p$ as the latter is equal to $\|T^j(x)\|_p$ which is the same as $\beta_{j,x}$. This is trivially true for $j = 0$. As shown earlier, for $1 \leq j < m-1$, $\beta_{j,u} = \|T^j(u)\|_p$ for all $u \in \mathcal{S}_n$, and hence $\beta_{j,y} = \beta_{j,x}$; i.e. the above statement holds for $1 \leq j < m$, and we get that

$$\overline{T}_a^j(y) = \beta_{j,y}T^j(y) + \beta_{j,y}\sum_{i=1}^{j}\beta_{j-i,y}^{-1}T^{i-1}(a) = \overline{T}_a^j(x) + \beta_{j,x}T^j(z). \tag{4.5}$$

We prove by induction on k that $\beta_{j,y} = \beta_{j,x} = \|T^j(x)\|_p$ and Eq. (5) is satisfied for all $1 \leq j < km$, $k \in \mathbb{N}$. We have already proven these for $k = 1$. Suppose for some

$k \in \mathbb{N}$, these hold for all j such that $(k-1)m \leq j < km$. Let $km \leq j < (k+1)m$. Recall that for all $j \in \mathbb{N}$, $\alpha_j(u) = \|T(\overline{T}_a^{j-1}(u))\|_p$, $u \in \mathcal{S}_n$, and Eq. (2) holds for any $x \in \mathcal{S}_n$ and $j \geq m$. As $\beta_{j,y}\beta_{j-m,y}^{-1} = \alpha_j(y)\ldots\alpha_{j-m+1}(y)$, from Eq. (2) and, also Eq. (5) which is assumed to hold for $(k-1)m \leq j < km$ by the induction hypothesis, we get for x, y, z as above and $km \leq j < (k+1)m$ that

$$\begin{aligned}
\beta_{j,y}\beta_{j-m,y}^{-1} &= \|T^{m-1}(a) + T^m(\overline{T}_a^{j-m}(y))\|_p \\
&= \|T^m[T^{-1}(a) + \overline{T}_a^{j-m}(x) + \beta_{j-m,x}T^{j-m}(z)]\|_p
\end{aligned}$$

Now using this, we get that

$$\beta_{j,y}\beta_{j-m,y}^{-1} = \|S[p^{-l}(T^{-1}(a) + \overline{T}_a^{j-m}(x)) + p^{l_1}\beta_{j-m,x}T^{j-m}(z)]\|_p = p^{-l},$$

as S is an isometry, $l_1 > l$ and $\|\beta_{j-m,x}T^{j-m}(z)\|_p < 1$ (see also Eq. (3)). Since $(k-1)m \leq j-m < km$, $\beta_{j,y} = p^{-l}\beta_{j-m,y} = \|p^l T^{j-m}(x)\|_p = \|T^j(x)\|_p$. Hence Eq. (5) holds for $km \leq j < (k+1)m$. Now by induction for all $j \in \mathbb{N}$, $\beta_{j,x} = \beta_{j,y}$ and Eq. (5) holds. Therefore, $\overline{T}_a$ is not distal. (Note that Eq. (5) also directly shows that $\overline{T}_a^{km}(y) - \overline{T}_a^{km}(x) = p^{k(l_1-l)}S^k(z) \to 0$ as $k \to \infty$). Now if $U \subset V$ is a neighbourhood of 0, then $U \cap H \neq \{0\}$ and hence (II) holds. $\square$

Observe that if $\overline{T}$ is not distal, then from Theorem 4.2.2 (II), we get that every neighbourhood of 0 in $\mathbb{Q}_p$ contains a nonzero a such that $\|T^{-1}(a)\|_p < 1$ and $\overline{T}_a$ is not distal. Now the following corollary is an easy consequence of Theorem 4.2.2.

Corollary 4.2.3. *For $T \in GL(n, \mathbb{Q}_p)$, $\overline{T}$ is distal if and only if there exists a neighbourhood V of 0 in $\mathbb{Q}_p^n$ such that for every $a \in V \setminus \{0\}$, $\|T^{-1}(a)\|_p < 1$ and $\overline{T}_a$ on $\mathcal{S}_n$ is distal.*

If T is distal, then $\overline{T}$ is also distal and Theorem 4.2.2 (I) and Corollary 4.2.3 holds for T. If $\overline{T}$ is distal, then for some $m \in \mathbb{N}$ and $l \in \mathbb{Z}$, $p^l T^m$ is distal.

Chapter 5

Distal actions on Sub_G

In this chapter, we consider the space Sub_G, the set of all closed subgroups of a topological group G endowed with the Chabauty topology. We first survey some known results. Then we shall study the distality of actions of automorphisms of G on Sub_G.

5.1 Chabauty Topology

This topology was introduced by Claude Chabauty [12] in 1950. Chabauty topology is defined as follows.

Definition 5.1.1. Let G be a locally compact topological group. Let Sub_G be the set of all closed subgroups of G. A sub-basis of *the Chabauty topology* on Sub_G is given by the sets of the following form

$$\mathcal{O}_1(K) = \{A \in \mathrm{Sub}_G \mid \mathrm{A} \cap \mathrm{K} = \emptyset\}, \quad \text{where} \quad K \subset G \text{ is compact, and}$$

$$\mathcal{O}_2(U) = \{A \in \mathrm{Sub_G} \mid \mathrm{A \cap U \neq \emptyset}\}, \quad \text{where} \quad U \subset G \text{ is open.}$$

Finite intersections $\mathcal{O}_1(U_1) \cap \cdots \cap \mathcal{O}_1(U_n) \cap \mathcal{O}_2(K_1) \cap \cdots \cap \mathcal{O}_2(K_m), m, n \geq 1$, constitute a basis for the *Chabauty topology* on $\mathrm{Sub_G}$. Observe that, $\mathcal{O}_2(K_1) \cap \cdots \cap \mathcal{O}_2(K_m) = \mathcal{O}_2(K_1 \cup \cdots \cup K_m)$, and finite union of compact sets is compact. Therefore one can also consider $\mathcal{O}_1(U_1) \cap \cdots \cap \mathcal{O}_1(U_n) \cap \mathcal{O}_2(K)$ as a basis for the *Chabauty topology.* Details of Chabauty topology follows from [12, 7, 3, 10] and [18]. For more history about this topology see [24].

We now list the following interesting properties of the space $\mathrm{Sub_G}$. We refer [7] for more explanation and proof of following properties.

Proposition 5.1.2. *Let G be a locally compact group, then the following hold:*

(1) $\mathrm{Sub_G}$ *is compact.*

(2) *If G is Hausdorff, then* $\mathrm{Sub_G}$ *is also Hausdorff.*

(3) *If G is metrizable, then* $\mathrm{Sub_G}$ *is metrizable.*

Throughout this chapter, we will assume that G is a locally compact metrizable group.

Lemma 5.1.3. *[see [7], page 161] Let G be as above. A sequence $\{\mathcal{H}_n\} \subset \mathrm{Sub_G}$ converges to $\mathcal{H} \in \mathrm{Sub_G}$ if and only if the following statements hold:*

(1) *If $g \in G$, there exists a subsequence $\{\mathcal{H}_{n_k}\}$ of $\{\mathcal{H}_n\}$ and $h_k \in \mathcal{H}_{n_k}$ such that $h_k \longrightarrow g$ in G, then $g \in \mathcal{H}$.*

(2) *For every* $h \in \mathcal{H}$, *there exists a sequence* $\{h_n\}_{n\in\mathbb{N}}$ *such that* $h_n \in \mathcal{H}_n$ *and* $h_n \longrightarrow h$ *in* G.

Following are some of known simple examples of Chabauty topology.

Example 5.1.4. *Consider* $G = \mathbb{R}$. *We know that all proper closed subgroups of* $\mathbb{R}$ *are of the form* $r\mathbb{Z}$, *for* $r \in \mathbb{R}$. $\mathrm{Sub}_{\mathbb{R}}$ *is homeomorphic to a compact interval* $[0, \infty]$ *(for more details see [4] and [10]).*

Example 5.1.5. *Consider* $G = \mathbb{Z}$. *The Chabauty space of* $\mathbb{Z}$, $\mathrm{Sub}_{\mathbb{Z}}$ *is homeomorphic to the subspace* $\{\frac{1}{n}\} \cup \{0\}$ *of* $[0, 1]$ *with the usual topology. (see [10]).*

Example 5.1.6. *For* $G = \mathbb{R}^2$, $\mathrm{Sub}_{\mathbb{R}^2}$ *is homeomorphic to* $\mathbb{S}^4$ *(Pourezza and Hubbard [26]).*

5.2 Actions of automorphisms of G on $\mathrm{Sub_G}$

Definition 5.2.1. Contraction Group: *Let* G *be a locally compact (Hausdorff) group with identity* e *and* T *an automorphism of* G. *For a* T*-invariant compact subgroup* K, *the* K*-contraction group of* T *is defined as*

$$C_K(T) = \{x \in G \mid T^n(x)K \to K\}.$$

For $K = \{e\}$, *the group* $C_{\{e\}}(T)$ *is denoted by* $C(T)$ *and is called the contraction group of* T.

We use this definition frequently in the proof of the main result in this section.

Let G be a locally compact metrizable group and let Aut(G) denote the group of automorphisms of G endowed with the compact open topology. There is a natural action of Aut(G) on $\mathrm{Sub}_{\mathrm{G}}$ defined as follows

$$\mathrm{Aut(G)} \times \mathrm{Sub_G} \to \mathrm{Sub_G},\ (T,H) \mapsto T(H);\ T \in \mathrm{Aut(G)},\ H \in \mathrm{Sub_G}.$$

This is a continuous group action. In this section, first we prove some elementary results and then compare the distality of T on $\mathrm{Sub_G}$ and that on G, for $T \in \mathrm{Aut(G)}$.

Lemma 5.2.2. *For G as above, the following hold:*

(1) *Let $H_n, L_n \in \mathrm{Sub_G}$ such that $H_n \subset L_n$ and $H_n \to H$, $L_n \to L$ in $\mathrm{Sub_G}$. Then $H \subset L$.*

(2) *Let $H, L_n \in \mathrm{Sub_G}$ be such that $H \subset L_n$ for all $n \in \mathbb{N}$, and $L_n \to L$ in $\mathrm{Sub_G}$. Then $H \subset L$.*

(3) *Let $H_0, L_0 \in \mathrm{Sub_G}$ be such that $H_0 \subset L_0$ and $T_n(H_0) \to H$, $T_n(L_0) \to L$, for $T_n \in \mathrm{Aut(G)}$. Then $H \subset L$.*

(4) *Let $T \in \mathrm{Aut(G)}$ and let $H, L \in \mathrm{Sub_G}$ be such that $T(H) = H$ and $H \subset L$. If L' be any limit point of $\{T^n(L)\}_{n\in\mathbb{Z}}$, then $H \subset L'$.*

(5) *If $T_n \to T$ in $\mathrm{Aut(G)}$, and $H_n \to H$ in $\mathrm{Sub_G}$, then $T_n(H_n) \to T(H)$.*

Proof. (1). Let $h \in H$. From the condition (2) of Lemma 5.1.3 there exists a sequence $\{h_n\}$ such that $h_n \in H_n$ for all n, and $h_n \to h$. As $H_n \subset L_n$ we have $h_n \in L_n$, for all n. Then from the condition (1) of Lemma 5.1.3, $h \in L$. Hence $H \subset L$.
(1) $\Rightarrow$ (2), as we can put $H_n = H$ in (1), for all n.

(1) $\Rightarrow$ (3), as we can take $H_n = T_n(H_0)$ and $L_n = T_n(L_0)$ in (1), for all n.

(2) $\Rightarrow$ (4) is obvious.

(5) follows from the continuity of the action of Aut(G) on $\mathrm{Sub_G}$. $\square$

Lemma 5.2.3. *For a closed normal subgroup H of G, let $\pi : G \to G/H$ be the canonical projection. Suppose $L_n \in \mathrm{Sub_G}$ be such that $H \subset \bigcap_{n\in\mathbb{N}} L_n$. Then the following hold:*

1. *If $L_n \to L$ in $\mathrm{Sub_G}$, then $\pi(L_n) \to \pi(L)$.*
2. *If $\pi(L_n) \to L'$, then $L_n \to \pi^{-1}(L')$.*

Proof. (1). Observe that as $\mathrm{Sub_{G/H}}$ is compact, $\{\pi(L_n)\}$ is relatively compact. Suppose $\pi(L_{n_k}) \to L'$ for some sequence $\{n_k\}$. First we show that $\pi(L) \subset L'$. From (2) of Lemma 5.1.3, for any $x \in L$, there exists a sequence $\{x_n\}$ such that $x_n \in L$ and $x_n \to x$ in G. This implies that $\pi(x_n) \to \pi(x)$ (as π is continuous). Therefore $\pi(x) \in L'$, and hence $\pi(L) \subset L'$.

Conversely, suppose $x' \in L'$. Again from (2) of Lemma 5.1.3, there exists a sequence $\{x'_k\} \subset \pi(L_{n_k})$ such that $x'_{n_k} \to x'$ in G/H. Then there exists a sequence $\{x_{n_k}\} \subset L_{n_k}$ such that $\pi(x_{n_k}) = x'_{n_k} \to x'$. There also exists a sequence $\{h_n\} \subset H$ such that $x_{n_k}h_{n_k} \to x$, for some $x \in G$. Now $x_{n_k}h_{n_k} \in L_{n_k}$ as $H \subset L_{n_k}$, for all k, and hence $x \in L$. Moreover, $\pi(x) = x' \in L'$, and hence $\pi(L) = L'$. Since this is true for all limit points of $\{\pi(L_n)\}$, we have that $\pi(L_n) \to \pi(L)$.

(2). As $\mathrm{Sub_G}$ is compact, for any sequence $\{L_n\} \subset \mathrm{Sub_G}$ there exists a convergent subsequence $\{L_{n_k}\}$ such that $L_{n_k} \to L$. Hence $\pi(L) = L'$, from (1) above. As $H \subset L_{n_k}$, for all k, $H \subset L$ (from (2) of Lemma 5.2.2), which implies that $L = \pi^{-1}(L')$, and hence $L_n \to \pi^{-1}(L')$.

□

Theorem 5.2.4. *Let G be a locally compact metrizable group, $T \in \mathrm{Aut(G)}$ and let H be a closed normal T-invariant subgroup of G. Let $\overline{T} \in \mathrm{Aut(G/H)}$ be the corresponding map defined as $\overline{T}(gH) = T(g)H$, for $g \in G$. If T acts distally on $\mathrm{Sub_G}$ then T acts distally on both $\mathrm{Sub_H}$ and $\mathrm{Sub_{G/H}}$.*

Proof. Suppose T acts distally on $\mathrm{Sub_G}$. Then the restriction of T on $\mathrm{Sub_H}$ is clearly distal. Now we show that $\overline{T}$ is distal. For $i = 1, 2$, let $H_i \in \mathrm{Sub_{G/H}}$ be such that $H \subset H_i$ and $\overline{T}^{n_k}(H_i) \to L$ in $\mathrm{Sub_{G/H}}$. This also implies that $T^{n_k}(\pi^{-1}(H_i)) \to \pi^{-1}(L)$, for $i = 1, 2$. Therefore $\pi^{-1}(H_1) = \pi^{-1}(H_2)$ (as T is distal on $\mathrm{Sub_G}$) which implies that $H_1 = H_2$. Therefore, $\overline{T}$ is distal. □

For any group G, let G^0 denotes the connected component of the identity e in G. Note that G^0 is a closed (normal) characteristics subgroup in G. There exists a unique maximal compact normal subgroup K in G^0, which is also characteristic in G and G^0/K is a Lie group. As observed in [28], every inner automorphism of G^0 acts distally on K^0. More generally, if $C(T)$ is closed, then T acts distally on K.

Theorem 5.2.5. *Let G be a locally compact metrizable group, $T \in \mathrm{Aut(G)}$ and let K be the maximal compact normal subgroup of G^0. If T is distal on $\mathrm{Sub_G}$, then T is distal on G/K^0. Moreover, if T acts distally on K^0 then T acts distally on G.*

Proof. Let T be distal on $\mathrm{Sub_G}$. Then by Theorem 5.2.4 it is distal on $\mathrm{Sub_{G/K^0}}$. Hence, without loss of any generality, we may assume that K as above is totally disconnected and show that T is distal. By Theorem 4.1 in [28], it is enough to show that both $C(T)$ and $C(T^{-1})$ are trivial.

Step-I. Suppose G is totally disconnected. Suppose that $C(T)$ is non trivial. Since $\overline{C(T)}$ is totally disconnected, there exists a neighbourhood basis of (proper) open compact normal subgroups $\{C_m\}_{m\in\mathbb{N}}$. Take $H = C_m$ for a fixed m. Since H is an open neighbourhood of the identity e in G, for $x \in C(T)$, $T^n(x) \in H$ for large n and $x = T^{-n}(T^n(x)) \in T^{-n}(H)$. Hence if $T^{-n_k}(H) \to L$ for a sequence $\{n_k\} \subset \mathbb{N}$, then $C(T) \subset L$. As L is closed, $\overline{C(T)} \subset L$. Since $\overline{C(T)}$ is T-invariant and $H \subset \overline{C(T)}$, $\overline{C(T)} = L$ and we have that $T^{-n}(H) \to \overline{C(T)}$ and hence T is not distal on $\mathrm{Sub}_{\overline{\mathrm{C(T)}}}$. Therefore, $C(T)$ is trivial.

For any locally compact group G, G/G^0 is totally disconnected. Let $\overline{T} : G/G^0 \to G/G^0$ be the natural projection. Then from Theorem 5.2.4, $\overline{T}$ is distal on $\mathrm{Sub}_{\mathrm{G/G^0}}$ and hence, from above, $C(\overline{T})$ is trivial. Therefore, $C(T) \subset G^0$.

Step-II. As K is totally disconnected and $T|_K$ acts distally on $\mathrm{Sub}_{\mathrm{K}}$, from Step-I we get that $C(T) \cap K = \{e\}$. Then by Proposition 4.3 of [28], $C(T)$ is closed and hence a simply connected nilpotent group. Now, let $N = C(T)$ and $N_1 = \overline{[N, N]}$. Then N/N_1 is homeomorphic to $\mathbb{R}^n$. Let T_1 be the projection of $T|_N$ on N/N_1. Then $C(T_1) = N/N_1$. Suppose T_1 has a real eigenvalue λ. Then $0 < |\lambda| < 1$ and there exists a subspace $M \approx \mathbb{R} \subset N/N_1$ such that $T_1(x) = \lambda x$ for all $x \in M$. It is easy to check that $T_1^n(\mathbb{Z}) \to \mathbb{R}$ as $n \to \infty$ in $\mathrm{Sub}_{\mathbb{R}}$ (cf. [4]). As T_1 is not distal on $\mathrm{Sub}_{\mathrm{N/N_1}}$, which is a contradiction.
Suppose T_1^m has a real eigenvalue for some $m \in \mathbb{N}$. Then arguing as above we get that T_1^m does not act distally on $\mathrm{Sub}_{\mathrm{N/N_1}}$ and hence on $\mathrm{Sub}_{\mathrm{N}}$, which is a contradiction.

Now suppose all the eigenvalues of T_1 are complex. Then T_1 keeps a two dimensional space $V \approx \mathbb{R}^2$ invariant such that $T_1|_V$ has a complex eigenvalue of the form $r(\cos\theta + i\sin\theta)$ where $0 < r < 1$ and θ is an irrational angle. Consider

$T_2 \in GL(2,\mathbb{R})$ such that $T_1|_V = T_2$ (under the isomorphism of $\mathbb{R}^2$ with V). Then $T_2 = rA_\theta$, where $A_\theta = AR_\theta A^{-1}$ for some $A \in GL(2,\mathbb{R})$ and R_θ is the rotation by the angle θ on $\mathbb{R}^2$. As observed in Chapter 2 earlier, A_θ generates a compact group i.e. $A_\theta^{n_k} \to \mathrm{Id}$ on $\mathbb{R}^2$, for some unbounded sequence $\{n_k\}$ as θ is irrational. Let $Z = \{(m,0) \mid m \in \mathbb{Z}\}$. Then $Z \in \mathrm{Sub}_{\mathbb{R}^2}$, passing to a subsequence if necessary, we have that $T_2^{n_k}(Z) \to M'$ for some M' in $\mathrm{Sub}_{\mathbb{R}^2}$. But $T_2^{n_k}(Z) = A_\theta^{n_k}(r^{n_k}Z)$. Now as $r^{n_k}(Z) \to \mathbb{R}\times\{0\} = \{(t,0) \mid t \in \mathbb{R}\}$ (cf. [4]) and $A_\theta^{n_k} \to \mathrm{Id}$. Therefore $T_2^{n_k}(Z) \to \mathbb{R}\times\{0\}$ in $\mathrm{Sub}_{\mathbb{R}^2}$ (by using (5) of Lemma 5.2.2). We also have that $T_2^{n_k}(\mathbb{R}\times\{0\}) = A_\theta^{n_k}(\mathbb{R}\times\{0\}) \to \mathbb{R}\times\{0\}$ (again by using (5) of Lemma 5.2.2). Hence T_1 does not act distally on $\mathrm{Sub}_{\mathrm{N/N_1}}$, a contradiction. Therefore, $C(T)$ is trivial in this case too.
Replacing T by T^{-1} and using the distality of T^{-1} on $\mathrm{Sub}_{\mathrm{G}}$, we conclude that $C(T^{-1})$ is trivial and hence T is distal on G/K^0.

The second assertion in the Theorem is obvious. □

The following corollary is the consequence of the above Theorem, which can be proven by using results in [28].

Corollary 5.2.6. *If G as above is connected and every inner automorphism of G acts distally on $\mathrm{Sub}_{\mathrm{G}}$, then G is distal, i.e. the conjugation action of G on G is distal.*

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