SEMICLASSICAL TREATMENT OF MULTI-MODE LASER SYSTEM

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by

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PREFACE

This dissertation entitled "Semiclassical Treatment of Multimode Laser System" has been carried out in the School of Theoretical and Environmental Science, Jawaharlal Nehru University, New Delhi. The work is original and has not been submitted in part or in full for any degree or diploma of any University.

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(Masroor Hasan)

CONTENTS

CHAPTER 1

INTRODUCTION

Lasers are one of the biggest achievements made in the second half of the twentieth century.

Lasers are quantum generator working in the optical region of the spectrum and so they may be called an extension of the maser principle to the optical domain.

The principle on which lasers operation is based is the amplification of electromagnetic oscillations by means of a forced or induced radiation of atoms and molecules. Einstein had predicted this kind of radiation as long as 1917 while studying the equilibrium between the energy of atomic systems and their radiation. Therefore, it will be not wrong to say that the history of creation of lasers begins just as early.

We present here a theoretical description of the operation of multimode laser oscillation. This approach is particularly suitable for gaseous lasers of the type suggested by Schallow and Townes[1] and first realised experimentally by Javan, Bannet and Harriott[2]

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1. A.L. Schallow and C.H. Townes, Phys. Rev. 112, 1940 (1958)
2. A. Javan, W.R. Bennet and D.R. Harriott, Phys. Rev. Letters 6, 106 (1961)

The equations obtained are also useful in describing some of the features of solid state Lasers.

We consider a classical electromagnetic field in a high - Q multimode cavity, acting on active medium consisting of collection of atoms. The atoms are described by the laws of quantum mechanics whereas electromagnetic field is described by Maxwell's equations. We do not consider the phenomenon of noise due to spontaneous emission, thermal density or quantum fluctuation. The intrinsic line width of the laser field, resulting from spontaneous emission etc. can only be obtained by taking into account the field quantization. Javan and Co-workers[1] have obtained a high degree of spectral purity which suggests that the exclusion of the phenomenon of noise in the semi-classical theory is a good approximation.

A macroscopic electric polarization P(r,t) is produced in the active medium due to interaction between the electromagnetic field and the atoms in the cavity. This macroscopic polarization acts as a source in the electromagnetic field in accordance with the Maxwell's equations. The amplitude and frequency of possible oscillations is obtained by the condition of self consistency (that the field produced should

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1. T.S. Jaseja, A. Javan and C.H. Townes, Phys. Rev. Letter **10**, 165 (1963)

be equal to the field assumed). Our calculation includes non-linear effect which describes the phenomenon of frequency pulling and mode competition.

We have assumed that only two atomic states 'a' and 'b' contribute to the laser action. We further make the simplification that the vector character (polarization) of the electric field is ignored. In our calculation cavity is employed with windows of the Brewsters angle type so that the optical configuration favours one plane of polarization.

The cavity of the Fabry Perot type used by Javan, Bennet and Harriott is not enclosed by a reflecting wall and so contains a continuum of modes. However Fox and Li[1] have shown that there are descrete sets of quasimodes for which the loss due to diffraction leakage is small. The cavity modes of highest Q are the even symmetric ones whose circular frequencies are given by

$$\Omega_n = \frac{\pi n c}{L}$$

where c is the velocity of light, L is the distance between the reflecting planes ($L \sim 100$ cm) and n is a large integer, of the order of 2×10^6. Fox and Li have shown that the mode

1. A.G. Fox and T. Li, Bell System Tech. J. 40, 61 (1961)

of the next highest Q differ by 1 mc/sec from the former modes. Our discussion will be specifically aimed at the mode of highest Q.

The basic model for the laser is schematized in the Fig. 1.1 we have N_o active 2 level atoms per unit volume all coupled to the laser field by an electric dipole moment. It can be imagined that each atom i is being coupled to its own "Pumping Reservoir" R_{pi}. Similarly each atom i is coupled to its own "Loss Reservoir" $R_{L,i}$ which describes the damping of the levels 'a' and 'b' due to (non laser) transition to other levels and to atomic collision. The laser field is also coupled to a 'Field Loss Reservoir' $R_{L,F}$ which describes the damping of the laser modes due to transmission through the semi-transparent mirror, diffraction losses, losses due to finite conductivity of the walls etc.

The laser system formed by the active atoms and the field (broken line box in the fig. 1) represents an 'open system' which interchanges energy with the pump and loss reservoirs. In the condition of stable oscillation above threshold, it is a system which is far from thermodynamical equilibrium; this is one of the reason why it is theoretically interesting.

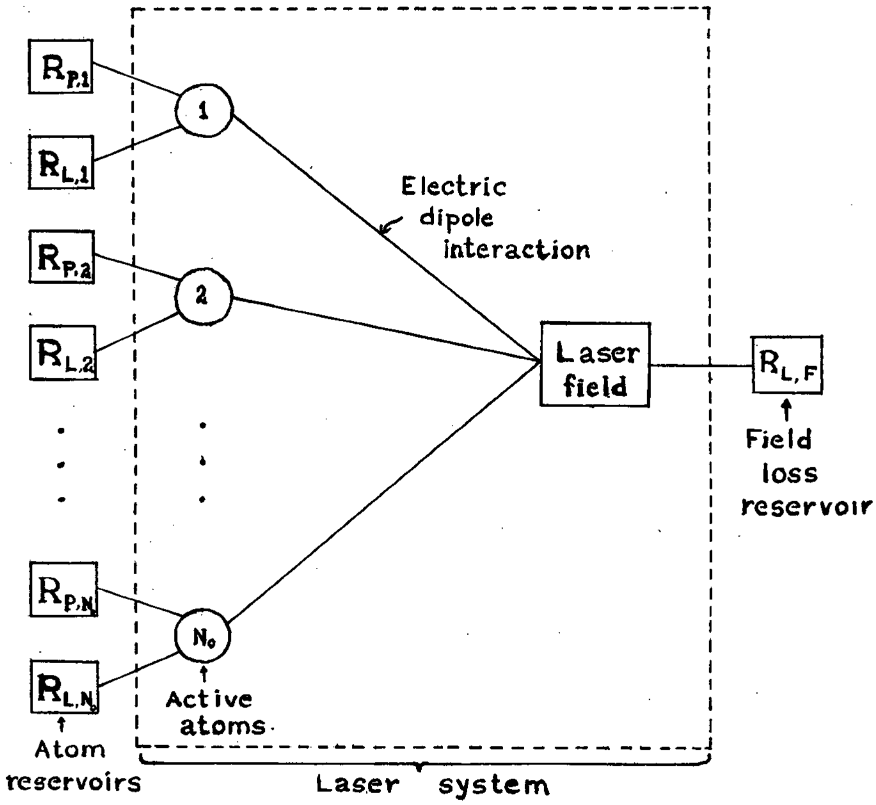

Figure. I. Model for the laser.

We have discussed the fundamental assumptions and the basic philosophy of the semiclassical theory in chapter 2 and then using the Maxwells equations we obtained the frequency and amplitude equations of the field in terms of polarization.

Chapter 3 describes how with the help of Schrodinger equation, microscopic polarization can be obtained in a two level atom system. At the end of this chapter we have introduced the density matrix formalism and deduced the equation of motion of density matrix, which is instrumental in our further calculation in the following chapters.

Macroscopic polarization has been calculated in chapter 4 with the help of density matrix. At the end of the chapter we have obtained a set of rate equations for diagonal and off diagonal elements of the density matrix.

The rate equations mentioned above have been solved in chapter 5 by assuming the population inversion density independent of time which helps in calculating macroscopic polarization $P(r,t)$. This chapter also determines the linear equation in E_n and conditions of laser oscillation; and then expression for critical population Inversion Density is obtained 'Frequency Pulling Effect' has been discussed at the end of the chapter.

Chapter 6 contains how rate equations of linear approximation are modified when we consider the time-dependence of population inversion density. We have presented the steady state solution of rate equations which defines the population inversion density in non-linear approximation. Next we have obtained the diagonal matrix element which helps in obtaining the field equation which is non-linear in E. Solving this non-linear field equation in the steady-state we get the expression for intensity of laser oscillation in one mode.

Mode competition phenomenon has been discussed in chapter 7. In this chapter amplitude equation has been derived assuming the probability of oscillation of more than one mode. At the end of the chapter we have discussed a simple case of two modes oscillation and shown how under certain circumstances one mode suppresses the other and when both modes oscillate simultaneously. The phenomenon of mode competition has also been explained with the help of phase-space diagram in linearized approximation.

We have concluded our discussion of the semi-classical theory in chapter 8 profiecting its success and failure in explaining certain important features of lasers. A more illuminating theory is required to explain the finer details of the laser problem, which is nothing but the quantum theory.

CHAPTER 2

ELECTROMAGNETIC FIELD

(a) BASIC ASSUMPTIONS AND PHILOSOPHY

In a laser an electromagnetic field is in resonance with the atomic transition between two levels enclosed in an optical cavity resonator. The resonance of the field with the atomic transition gives rise to stimulated emission; the emitted radiation is again in resonance and gives rise to further transition in other atoms and in this manner a kind of chain reaction or photon avalanche starts.

The field in a cavity is characterised by a discrete set of eigen modes. The atoms taking part in laser operation are excited by a pumping mechanism. This pumping causes electronic transition, to a large number of excited states of which same pairs will be in resonance with the cavity modes. These transition will amplify the field if the population of atoms in the upper level is more than the population in the lower level. In thermal equilibrium, lower level is more populated than the upper level. Hence the reverse situation which is the pre-requisite of laser operation is called "Population Inversion".

Another important requirement is that in a cavity there are large number of modes competing with the energy in the frequency range of atomic transition. To get the laser action, this is reduced to a small number by employing, instead of a cavity a laterally open system with mirrors at the ends.

When the amplification by the resonant transition exactly compensates the losses in the cavity due to imperfect reflection etc., the laser oscillates in a steady-state. If the gain is higher than the loss, the intensity increases untill a new stable state is reached. Thus it is important to consider the non-linear properties of the laser, because the output intensity is determined by the saturation behaviour.

Let us consider ansemble of excited two level atoms, upper state $|\psi_a\rangle$ lower state $|\psi_b\rangle$ placed between the mirrors of a laser cavity at time $t = 0$. We assume that at this initial time there exists a small electric field in the cavity. The atoms will respond to this electric-field and begin to oscillate as tiny dipoles. These atomic dipoles add up to give a macroscopic polarization per unit volume. The macroscopic dipole moment now drives the field i.e. acts as a source of radiation. The theory of laser action as formulated by Lamb[1] may be summerised in the following three steps.

1. W.E. Lamb, Phys. Rev., 134, A 1429 (1964)

1. Assuming an initial field $E(r,t)$ acting on an atom (say the ith atom) injected into the laser cavity, we calculate the atomic polarization $< p_i >$ accoring to the laws of quantum mechanics.

2. These atomic dipoles add up to give a macroscopic dipole moment per unit volume

$$P(r,t) = \sum_{i=1}^{N} < p^i(r,t) >$$

where N is the number of lasing atoms per unit volume in the cavity at time t. If the excitation is uniform we may write

$$P(r,t) = N < p(r,t) >$$

3. This polarization $P(r,t)$ drives the laser field according to Maxwells equation.

We assume each atom envolves in a field prepared for it by all the other atoms, then look for the field produced by many such envolving atoms. In this way the laser problem is similar to that of a fearomagnet in which each spin sees a mean magnetic field due to all the three spins and aligns itself accordingly, thus contributing to the average magnetic field. There each laser atom interacts with the electromagnetic field produced by all the other atoms that have contributed to the field via stimulated emission.

Both semi-classical and quantum theory of the laser are based on a number of assumptions. These are listed below.

1. Two Level Atoms

We assume only two level systems take part in the transition that gives rise to laser radiation.

2. Electric Dipole Approximation

The interaction between the two level atoms and the radiation field is treated in the electric dipole approximation. This is justified by the fact that optical wave length are much greater than atomic dimension.

3. Absence of Direct Interatomic Interaction

The direct interaction among the atoms of the active medium is neglected. They interact via the common radiation field.

4. Rotating Wave Approximation

In the expression of the type

$$\frac{e^{i(\omega_n-\omega_o)t-\gamma t}}{\omega_o-\omega_n-i\gamma} + \frac{e^{-i(\omega_n-\omega_o)t-\gamma t}}{\omega_o+\omega_n-i\gamma}$$

where ω_o, γ correspond to atomic frequency and line width, ω_n the mode frequency, the second term (anti-resonant) is neglected as compared to the first term (resonant) because the mode is close to resonance $|\omega_o-\omega_n| \leqslant \gamma$ whereas $(\omega_o+\omega_n) \gg \gamma$. The neglected term is both small (due to denominator) and rapidly oscillating. This approximation is known as 'Rotating Wave Approximation'.

(b) FREQUENCY AND AMPLITUDE EQUATIONS

The intensity of the electromagnetic field in the laser is very high there it is justified to consider the electromagnetic field classically; this is in agreement with the correspondence principle which states that quantum mechanics goes over to classical mechanics for large quantum number. It is very helpful for many purposes to treat the electromagnetic field in classical terms i.e. it is not quantized and treat the atoms quantum mechanically.

Condier the equations for the field due to a given macroscopic polarization $\bar{P}(r,t)$. The electromagnetic field in a cavity is determined from Maxwells equations

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \tag{1.1}$$

$$\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \tag{1.2}$$

$$\text{div}\, \bar{B} = 0$$

$$\text{div}\, \bar{D} = 0$$

where

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

$$\vec{B} = \mu_o \vec{H} \qquad (1.3)$$

$$J = \sigma \vec{E}$$

σ is the equivalent ohmic conductivity which is introduced here as a purely phenomenological parameter to represent the damping of the laser intensity which arises from the field losses due to imperfect reflections in the cavity and mirrors and other causes.

Equation (1.2) may be written as

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \frac{\partial}{\partial t} (\varepsilon_o \vec{E} + \vec{P})$$

or

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon_o \frac{\partial}{\partial t} \vec{E} + \frac{\partial}{\partial t} \vec{P} \qquad (1.4)$$

Equation (1.1) may be written as

$$\vec{\nabla} \times \vec{E} = - \frac{\partial}{\partial t} \mu_o \vec{H} = - \mu_o \frac{\partial}{\partial t} \vec{H} \qquad (1.4a)$$

Taking the curl of equation (1.4a) we obtain

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \times (- \mu_o \frac{\partial}{\partial t} \vec{H})$$

$$= - \mu_o \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H}) \qquad (1.5)$$

From above

$$\bar{\nabla} \times \bar{H} = \sigma\bar{E} + \varepsilon_o \frac{\partial E}{\partial t} + \frac{\partial}{\partial t} \bar{P}$$

or

$$\frac{\partial}{\partial t} (\bar{\nabla} \times \bar{H}) = \frac{\partial}{\partial t} (\sigma\bar{E} + \varepsilon_o \frac{\partial \bar{E}}{\partial t} + \frac{\partial}{\partial t} \bar{P})$$

or

$$\frac{\partial}{\partial t} (\bar{\nabla} \times \bar{H}) = \frac{\partial}{\partial t} \sigma\bar{E} + \varepsilon_o \frac{\partial^2 \bar{E}}{\partial t^2} + \frac{\partial^2 \bar{P}}{\partial t^2} \qquad (1.6)$$

From equation (1.5)

$$\frac{\partial}{\partial t} (\bar{\nabla} \times \bar{H}) = - \frac{\bar{\nabla} \times (\bar{\nabla} \times \bar{E})}{\mu_o} \qquad (1.7)$$

From equation (1.6) and (1.7)

$$- \frac{\bar{\nabla} \times (\bar{\nabla} \times \bar{E})}{\mu_o} = \frac{\partial}{\partial t} \sigma \bar{E} + \varepsilon_o \frac{\partial^2 \bar{E}}{\partial t^2} + \frac{\partial^2}{\partial t^2} \bar{P}$$

or

$$- \bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = \sigma\mu_o \frac{\partial \bar{E}}{\partial t} + \mu_o\varepsilon_o \frac{\partial^2 \bar{E}}{\partial t^2} + \mu_o \frac{\partial^2 \bar{P}}{\partial t^2}$$

or

$$\sigma \mu_o \frac{\partial \bar{E}}{\partial t} + \mu_o\varepsilon_o \frac{\partial^2 \bar{E}}{\partial t^2} + \bar{\nabla} \times (\bar{\nabla} \times H) = - \mu_o \frac{\partial^2 \bar{P}}{\partial t^2}$$

or

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) + \mu_o\sigma \frac{\partial \bar{E}}{\partial t} + \mu_o\varepsilon_o \frac{\partial^2 \bar{E}}{\partial t^2} = - \mu_o \frac{\partial^2 \bar{P}}{\partial t^2} \qquad (1.8)$$

Since solution of the laser field will be practically menochromatic, we can write

$$\frac{\partial^2 \bar{P}}{\partial t^2} \simeq -\omega^2 \bar{P}$$

where ω is the mean angular frequency of oscillation of $\bar{P}$ which be latter identified with frequency ω_n of the laser field. This approximation essentially means that the polarization $\bar{P}$ contains parts which oscillate with frequency within a narrow range only. Let us confine our attention to axial mode and expand the electric field in terms of a complete set of such modes.

$$\bar{E}(z,t) = \sum A_n(t)\, \bar{u}_n(z)$$

$$= \sqrt{\frac{2}{L}}\, A_n(t) \sin(k_n z) \qquad (1.9a)$$

where

$$k_n = \frac{n\pi}{L} = \frac{\Omega_n}{c}$$

Here L is the laser length ($\approx$ 100 cm), Ω_n is the eigen frequency of the cavity mode, and the coordinate system introduced is one with the laser axis along the z axis. The electromagnetic field is transverse in nature, hence we have two modes for each k_n. But for convenience we assume that one mode is suppressed by Brewster's windows and only one mode for each k_n is considered. Similarly we can expand polarization $\bar{P}(z,t)$ in the cavity eigen-functions as

$$\bar{P}(z,t) = \sum \bar{P}_n(t)\, \bar{u}_n(z) \tag{1.9}$$

where $\bar{u}_n(z)$ satisfies

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{u}_n(z)) - \mu_o \varepsilon_o \Omega_n^2\, \bar{u}_n(z) = 0 \tag{1.10}$$

and the boundary conditions.

Substituting equation (1.9a) and (1.9) in equation (1.8) and using equation (1.10) we get

$$\frac{d^2A_n(t)}{dt^2} + \frac{\sigma}{\varepsilon_o} \frac{dA_n(t)}{dt} + \Omega_n^2 A_n(t) = \frac{\omega^2}{\varepsilon_o} P_n(t) \tag{1.11}$$

where the driving force $P_n(t)$ is the projection of the inhomogeneous source term $\bar{P}(z,t)$ on the mode n

$$P_n(t) = \int_0^L P(z,t)\, u_n(z)\, dz \tag{1.11a}$$

We see that the unknown mode amplitudes $A_n(t)$ in equation (1.11) obey the equation of motion for the driven damped harmonic oscillator.

Consider the simple time dependence of $A_n(t)$ i.e.

$$A_n(t) = A_n e^{i\omega t}$$

substituting the above in equation (1.11) the L.H.S. is obtained as equal to

$$-\omega^2 e^{i\omega t} + \frac{\sigma}{\varepsilon_o} i\omega e^{i\omega t} + \Omega_n^2 e^{i\omega t}$$

$$= e^{i\omega t} \left(-\omega^2 + i\omega \frac{\sigma}{\varepsilon_o} + \Omega_n^2 \right)$$

The factor

$$-\omega^2 + i\omega \frac{\sigma}{\varepsilon_o} + \Omega_n^2 = (\Omega_n + \omega)(\Omega_n - \omega) - i \frac{\sigma}{\varepsilon_o} \omega \qquad (1.12)$$

Now resonance interaction can take place when $\omega \approx \Omega_n$.

Thus expression (1.12) becomes

$$\approx 2\omega(\Omega_n - \omega) + i\omega \frac{\sigma}{\varepsilon_o}$$

$$= 2\omega\left[(\Omega_n - \omega) + i \frac{\sigma}{2\varepsilon_o}\right] \qquad (1.13)$$

The half-width of the resonance is given by

$$\Delta\omega = \frac{\sigma}{\varepsilon_o}$$

The cavity Q value is defined in the usual way

$Q \equiv \Omega_n$/fraction of stored energy dissipated per unit time

Since the energy decays e^{-t/τ_n}, where τ_n is the mode life time,

$$Q_n = \Omega_n \tau_n = \frac{\Omega_n}{\Delta\Omega_n} = \frac{\omega_n}{\Delta\omega_n}$$

Hence

$$Q = \frac{\omega}{\Delta\omega} = \frac{\omega}{\sigma/\varepsilon_o} = \frac{\omega\varepsilon_o}{\sigma} \tag{1.14}$$

equation (1.14) is introduced into equation (1.11) to eliminate σ and result obtained is

$$\frac{d^2 A_n(t)}{dt^2} + \frac{\omega}{Q_n}\frac{d A_n(t)}{dt} + \Omega_n^2 A_n(t) = \frac{\omega^2}{\varepsilon_o} P_n(t) \tag{1.15}$$

equation (1.15) will become a differential equation for the determination of $A_n(t)$ after imposing the self consistency requirement. The total output energy cannot exceed the energy supplied by optical pumping and a saturation effect is obtained. A steady state solution corresponding to the stable amplitude of oscillation can only be obtained from a non-linear theory.

This non-linear differential equation can be solved by a classical method - "Method of slowly varying amplitudes and phases" due to Krylov and Bogoliubov. The following "Ansatz" is made for the ~~forlution~~ solution

$$A_n(t) = E_n(t) \cos (\omega_n t + \varphi_n(t)) \tag{1.16}$$

Resolving the projection of the driving polarization $P_n(t)$ into a component in phase with the electric field and a component $\pi/2$ out of phase we get ansatz

$$P_n(t) = C_n(t)\cos[\omega_n t + \varphi_n(t)]$$

$$+ S_n(t)\sin[\omega_n t + \varphi_n(t)] \qquad (1.17)$$

where $E_n(t)$, $\varphi_n(t)$, $C_n(t)$ and $S_n(t)$ are slowly varying compared to $\cos\omega_n t$ and $\sin\omega_n t$. The ansatz (1.16) and (1.17) are introduced in (1.15) and since $E_n(t)$, $\varphi_n(t)$ are slowly varying

$$\ddot{E}\,,\ \ddot{\varphi}\,,\ \dot{\varphi}\dot{E}\,,\ \frac{\dot{E}}{Q_n}\,,\ \frac{\dot{\varphi}E}{Q_n}$$

can be neglected.

Since

$$A_n(t) = E_n(t)\cos(\omega_n t + \varphi_n(t))$$

Therefore

$$\frac{d\,A_n(t)}{dt^2} = -(\omega_n + \dot{\varphi}_n(t))E_n\sin(\omega_n t + \varphi_n(t)) +$$

$$\dot{E}_n\cos(\omega_n t + \varphi_n(t)) \qquad (1.18)$$

and

$$\frac{d^2A_n(t)}{dt^2} = \cos(\omega_n t + \varphi_n(t))[\underline{\ddot{E}_n} - E_n(\omega_n^2 + 2\omega_n\dot{\varphi}_n(t) + \dot{\varphi}_n^2(t)]$$

$$-\sin(\omega_n t + \varphi_n(t))[\dot{E}_n 2(\omega_n + \underline{\dot{\varphi}_n} t) + \underline{E_n\ddot{\varphi}_n(t)}]$$

neglecting the underlined terms in the above relation we

we get

$$\frac{d^2A_n(t)}{dt^2} = \cos(\omega_n t+\varphi_n(t))[-E_n(\omega_n^2 + 2\omega_n\dot\varphi_n(t)-\dot\varphi_n^2(t))]$$

$$- \sin(\omega_n t+\varphi_n(t))[2\dot E_n\omega_n] \qquad (1.19)$$

Now we put equations (1.17), (1.18) and (1.19) in equation (1.15) and obtain

$$\cos(\omega_n t + \varphi_n(t))[-E_n(\omega_n^2+2\omega_n+\dot\varphi_n(t)+\dot\varphi_n^2(t)]-\sin(\omega_n t+\varphi_n(t)) \times$$

$$(2\dot E_n\omega_n)+ \underline{\frac{\omega}{Q_n} \dot E_n\cos(\omega_n t+\varphi_n(t))} - \frac{\omega}{\underline{Q_n}} (\omega_n+\underline{\dot\varphi_n(t)})E_n \times$$

$$\sin(\omega_n t+\varphi_n(t)) + \Omega_n^2E_n\cos(\omega_n t+\varphi_n(t))$$

$$= \frac{\omega^2}{\varepsilon_o} [C_n(t) \cos(\omega_n t+\varphi_n(t))+S_n(t)\sin(\omega_n t+\varphi_n(t))]$$

Neglecting the underlined terms, and rearranging we get

$$\cos(\omega_n t+\varphi_n(t))[-E_n\{(\omega_n^2+2\omega_n\dot\varphi_n(t)+\dot\varphi_n^2(t))- \Omega_n^2\}] -$$

$$\sin(\omega_n t+\varphi_n(t))[2\dot E_n\omega_n+ \frac{\omega\,\omega_n}{Q_n} E_n]$$

$$= \frac{\omega^2}{\varepsilon_o} C_n(t)\cos(\omega_n t+\varphi_n(t))+ \frac{\omega^2}{\varepsilon_o} S_n(t)\sin(\omega_n t+\varphi_n(t))$$

(1.20)

Equating the coefficient of sin in the above equation we get

$$\frac{\omega^2}{\varepsilon_o} S_n(t) = - 2\dot E_n\omega_n - \frac{\omega\,\omega_n}{Q_n} E_n$$

or

$$\frac{\omega}{\varepsilon_o} S_n(t) = -2\dot E_n- \omega \frac{E_n}{Q_n}$$

or

$$\dot{E}_n(t) + \frac{1}{2}\frac{\omega}{Q_n} E_n(t) = -\frac{1}{2}\frac{\omega}{\varepsilon_o} S_n(t) \qquad (1.21)$$

equation (1.21) is obtained by putting $\omega_n \approx \omega$, because only that part of P_n which oscillate with $\omega_n \approx \omega$ will be important as the distance Δ between the cavity eigen-function is much larger than the width of the cavity resonance, $\Delta\omega \leqslant \Delta$ - Equating the coefficient of cos in the equation (1.20) we get

$$\frac{\omega_n^2}{\varepsilon_o} C_n(t) = - E_n(\omega_n^2 + 2\omega_n \dot{\varphi}_n(t) + \dot{\varphi}_n^2(t) - \Omega_n)$$

or

$$E_n[\Omega_n^2 - (\omega_n + \dot{\varphi}_n(t))^2] = \frac{\omega_n^2}{\varepsilon_o} C_n(t)$$

or

$$E_n(\Omega_n + \omega_n + \dot{\varphi}_n(t))(\Omega_n - \omega_n - \dot{\varphi}_n(t)) = \frac{\omega_n^2}{\varepsilon_o} C_n(t)$$

The frequency of oscillation ω_n is very close to the cavity eigen frequency Ω_n and we get

$$(\Omega_n + \omega_n + \dot{\varphi}_n(t)) \approx 2\omega_n$$

Hence

$$E_n(\Omega_n - \omega_n - \dot{\varphi}_n(t)) = \frac{1}{2}\frac{\omega_n^2}{\varepsilon_o} C_n(t)$$

or

$$E_n(\Omega_n - \omega_n - \dot{\varphi}_n(t)) = \frac{1}{2}\frac{\omega_n^2}{\varepsilon_o} C_n(t)$$

or

$$(\omega_n + \dot{\varphi}_n(t) - \Omega_n)E_n = -\frac{1}{2}\frac{\omega_n}{\varepsilon_o} C_n(t) \qquad (1.22)$$

Thus we get the two conditions

$$\dot{E}_n(t) + \frac{1}{2}\frac{\omega_n}{Q_n} E_n(t) = -\frac{1}{2}\frac{\omega_n}{\varepsilon_o} S_n(t) \qquad (1.21)$$

$$[\omega_n+\dot{\varphi}_n(t) - \Omega_n]E_n(t) = -\frac{1}{2}\frac{\omega_n}{\varepsilon_o} C_n(t) \qquad (1.22)$$

These equations determine the amplitude E_n and the frequency ω_n when P_n is known.

CHAPTER 3

MICROSCOPIC POLARIZATION

After finding the frequency equation (1.22) and amplitude equation (1.21) we will proceed to find the microscopic polarization with the help of Schrödinger equation for a 2 level systems.

According to our assumption (1) we may represent an active atom by a two level system (Fig. 3.1)

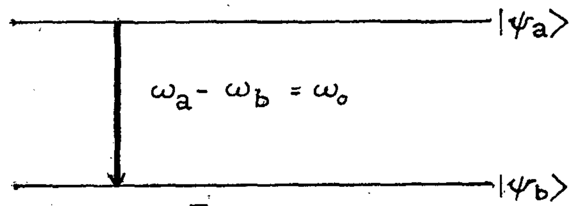

Fig. 3.1

Let Ha = Hamiltonian for a free atom

The stationary eigen functions of the two states $|\psi_a\rangle$ and $|\psi_b\rangle$ respectively satisfy

$$H_A \psi_a(x) = \hbar\, \omega_a \psi_a(x)$$
$$H_A \psi_b(x) = \hbar\, \omega_b \psi_b(x) \qquad (2.1)$$

together with

$$\int \psi_a \psi_a^* \, d\tau = \int \psi_b \psi_b^* \, d\tau = 1$$
$$\int \psi_a \psi_b \, d\tau = 0 \qquad (2.2)$$

and

$$\omega_a - \omega_b = \omega_o \tag{2.3}$$

The corresponding time dependent wave function is given by

$$\begin{aligned} \psi_a(x,t) &= \psi_a(x)\ \exp(-i\omega_a t) \\ \psi_b(x,t) &= \psi_b(x)\ \exp(-i\omega_b t) \end{aligned} \tag{2.4}$$

In the Weiss Kopf-Wigner approximation[1] the decay level 'a' and 'b' takes place exponentially

$$\begin{aligned} \psi_a(x,t) &= \psi_a(x) \exp[-i(\omega_a - i\ \tfrac{\gamma_a}{2})t] \\ \psi_b(x,t) &= \psi_b(x) \exp[-i(\omega_b - i\ \tfrac{\gamma_b}{2})t] \end{aligned} \tag{2.5}$$

where

$$\frac{1}{\gamma_a} = \tau_a \ (= \text{life time})$$

Now we introduce the atomic loss reservior by phenomenelogical replacement

$$\begin{aligned} \omega_a &\rightarrow \omega_a - i\ \frac{\gamma_a}{2} \\ \omega_b &\rightarrow \omega_b - i\ \frac{\gamma_b}{2} \end{aligned} \tag{2.6}$$

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1. W. Heitler, The Quantum Theory of Radiation, 3rd Edition Oxford University Press (1954) P. 182.

Hamiltonian of atom interacting with the laser field may be written in the dipole approximation

$$H = H_A + \hbar V \tag{2.7}$$

where

$$\hbar V = -exE(r,t) \tag{2.8}$$

and r is a fixed point in the atom.

Let at the initial time $t = t_o$ the atom is pumped to level a. Then

$$\psi(x,t_o) = \psi_a(x) \tag{2.9}$$

Under the influence of perturbation ($\hbar V$) transition to level b is possible due to induced emission and so the wave function at a latter time t becomes

$$\psi(x,t_o) = a(t)\psi_a(x) + b(t)\psi_b(x) \tag{2.10}$$

This state may be represented by a column vector

$$|\psi>_t = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

Time dependent Schrodinger equation is

$$i\hbar \frac{\partial\psi}{\partial t} = H\psi$$

or

$$i\hbar \frac{\partial\psi}{\partial t} = (H_A+\hbar V)\psi$$

Substituting for ψ from (2.10) we get

$$i(\dot{a}\psi_a + \dot{b}\psi_b) = a\omega_a\psi_a + b\omega_b\psi_b + aV\psi_a + bV\psi_b$$

Taking the vector product of above with ψ_a^* and ψ_b^* respectively and using the relation (2.2) we get

$$i\dot{a} = a\omega_a + bV_{ab} + aV_{aa}$$

and

$$i\dot{b} = b\omega_b + aV_{ba} + bV_{bb}$$

Thus we have

$$\begin{aligned} i\dot{a} &= \omega_a a + V_{ab} b \\ i\dot{b} &= V_{ba} a + \omega_b b \end{aligned} \qquad (2.11)$$

where

V_{ab} is the matrix element $\int \psi_a^* V \psi_b d\tau \equiv V_{ba}^* = -E(r,t)\frac{d}{\hbar}$

Since

$$\hbar V = -exE(r,t)$$

$$V = -\frac{ex}{\hbar}E(r,t)$$

Now

$$\int \psi_a^* V \psi_b \, d\tau = -\frac{ex}{\hbar} \int \psi_a^* \, x \, \psi_b \, d\tau$$

or

$$V_{ab} = -\frac{E}{\hbar} \int \psi_a^* \, ex \, \psi_b \, d\tau$$

$$= - E \cdot \frac{d}{\hbar}$$

where

$$d = \int \psi_a^* \, ex \, \psi_b \, d\tau = e \int \psi_a^* \, x \, \psi_b d\tau = e < x_{ab} >$$

and is known as "Transition Dipole Moment".

In deducing (2.11) we have made use of the fact $V_{aa} = V_{bb} = 0$ which means that the expectation value of electric dipole moment in a stationary state vanishes. This becomes obvious due to the fact that dipole moment operater has negative parity whereas, due to definite parity of a stationary state, the probability distribution has a positive parity.

The expectation value of the electric dipole moment in the state $\psi(x,t)$ $(= a(t)\psi_a(x)+b(t)\psi_b(x))$ is given by

$$< p > = e < \psi|x|\psi >$$

$$= e \int \psi^* x\psi d\tau = e[\int (a^* \psi_a^* + b^* \psi_b^*) x (a\psi_a + b\psi_b) d\tau]$$

$$= e[\int aa^* \psi_a^* x\psi_a d\tau + \int b^* a \psi_b^* x\psi_a d\tau +$$

$$a^* b \int \psi_a^* x \, \psi_b d\tau + b^* b \int \psi_b^* \, x \, \psi_b d\tau]$$

$$= e[a^*b \int \psi_a^* \, x \, \psi_b d\tau + ab^* \int \psi_b^* \, x \, \psi_a \, d\tau$$

$$= a^*b \; e \int \psi_a^* \, x \, \psi_b d\tau + ab^* e \int \psi_b^* \, x \, \psi_a \, d\tau$$

$$= a^*bd + ab^*d$$

$$= d(a^*b + ab^*)$$

Therefore

$$\langle p \rangle = d(a^*b + ab^*) \tag{2.12}$$

Now equation (2.11) can be written in the following matrix form.

$$i \frac{d}{dt} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \omega_a - i \frac{\gamma_a}{2} & V_{ab} \\ V_{ba} & \omega_b - i \frac{\gamma_b}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.13}$$

Here we have introduced damping terms γ_a and γ_b by replacing ω_a and ω_b according to (2.6). We write (2.13) in the following form

$$i \frac{d}{dt} \begin{pmatrix} a \\ b \end{pmatrix} = [H - i(\frac{\Gamma}{2})] \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.14}$$

where

$$H = H_o + V$$

$$= \begin{pmatrix} \omega_a & 0 \\ 0 & \omega_a \end{pmatrix} + \begin{pmatrix} 0 & V_{ab} \\ V_{ba} & 0 \end{pmatrix}$$

and

$$\Gamma = \begin{pmatrix} \gamma_a & 0 \\ 0 & \gamma_b \end{pmatrix}$$

Let us now define a density matrix (operator)

$$\rho(t) = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a^* & b^* \end{pmatrix}$$

$$= \begin{pmatrix} aa^* & ab^* \\ ba^* & bb^* \end{pmatrix} = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}$$

From the property of a density matrix we know that

$$T_r\rho(t) = 1$$

i.e.

$$|a|^2 + |b|^2 = 1$$

and this is only time when $\Gamma = 0$. $\rho(t)$ may be written in the matrix element form as

$$\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix} = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}$$

Thus

$$a^*b = \rho_{ba} \text{ and } ab^* = \rho_{ab} \tag{2.15}$$

Substituting from (2.15) in equation (2.12) we get the dipole moment in terms of the element of the density matrix

$$\langle p \rangle = d(\rho_{ba} + \rho_{ab})$$

$$= d(\rho_{ab} + \rho^*_{ab})$$

because ρ is a Hermitian matrix.

Thus

$$\langle p \rangle = d(\rho_{ab} + \rho^*_{ab}) \tag{2.16}$$

We have

$$\rho = \begin{pmatrix} a \\ b \end{pmatrix} (a^* \quad b^*)$$

Therefore

$$\frac{d}{dt}\rho = \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} (a^* \quad b^*) + \begin{pmatrix} a \\ b \end{pmatrix} (\dot{a}^* \quad \dot{b}^*)$$

or

$$i \frac{d}{dt} \rho = i \begin{pmatrix} \dot{a} \\ \\ \dot{b} \end{pmatrix} (a^* \quad b^*) + \begin{pmatrix} a \\ \\ b \end{pmatrix} (\dot{a}^* \quad \dot{b}^*)$$

Using (2.14) for the first term and using Hermitian conjugate of (2.14) for the second term in the above equation we get

$$i \frac{d}{dt} \rho = (H - i \frac{\Gamma}{2})\rho - \rho(H + i \frac{\Gamma}{2})$$

$$= H\rho - \rho H - \frac{i}{2} (\frac{\Gamma}{2} \rho + \rho \frac{\Gamma}{2})$$

$$= [H,\rho] - \frac{i}{2} \{ \rho,\rho \}$$

Thus we get the equation of motion for ρ

$$i \frac{d}{dt} \rho = [H,\rho] - \frac{i}{2} \{\Gamma,\rho\} \qquad (2.17)$$

where $\{\Gamma,\rho\}$ stands for the anti-commutator.

This equation of motion is very instrumental in finding the macroscopic polarization and hence the amplitude of the laser field, which will be shown in the following chapters.

CHAPTER 4

MACROSCOPIC POLARIZATION

Now we will compute the macroscopic polarization P by statistical summation of the microscopic dipole moments of all active atoms.

We have just described in the previous chapter that a single atom pumped to a level 'a' at a time to is described at time t by a density matrix (operator) $\rho(a, t_o, t)$ which is the solution of the equation of motion (2.17) and which satisfies the initial condition

$$\rho(a, t_o, t) = \rho(a) \qquad (3.1)$$

where

$$\rho(a) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

the atom is state a.

The corresponding contribution to the expectation value of the dipole moment at time t is given by

$$< p > = T_r[\rho(t_o, t), \hat{p}] \qquad (3.2)$$

from the property of density matrix where $\hat{p}$ is the operator corresponding to the transition dipole moment

$$d\sigma_1 = e < x_{ab} > \sigma_1$$

σ_1 is the Pauli matrix.

The above equation (3.2) may be written employing (2.16)

$$< p > = d[\rho_{ab}(a,t_o,t) + \rho^*_{ab}(a,t_o,t)] \qquad (3.3)$$

Different atoms i are pumped to level 'a' at different time t_{oi}. Thus the macroscopic polarization is given by

$$P(r,t) = \sum_{t_{oi} < t} T_r[\rho(a,t_{oi},t)\ \hat{p}] \qquad (3.4)$$

where sum is extended over all active atoms per unit volume which has been pumped to level 'a' upto time t (i.e. for all $t_{oi} < t$).

Writing

$$\sum \rho(a,t_{oi},t) = \rho(a,t)$$

equation (3.4) becomes

$$P(r,t) = T_r[\rho(a,t)\ \hat{p}] \qquad (3.5)$$

Let us define $\lambda_a(t_o)\,dt_o$ = average number of atoms pumped to level 'a' during t_o– t_o+dt_o per unit volume over r. This is known as average pumping rate density we have omitted the r dependence in the rotation.

Then we can write

$$\rho(a,t) = \sum_{t} \rho(a,t_{oi},t)$$

as

$$\rho(a,t) = \int_{-\infty} \lambda_a(t_o)\, \rho(a,t_o,t)\, dt_o \qquad (3.6)$$

Taking into account the initial condition (3.1) and differentiating (3.6) w.r. t_o time we get the equation of motion for $\rho(a,t)$ as

$$i \frac{\partial}{\partial t} \rho(a,t) = i\, \lambda_a(t)\, \rho(a) + \frac{\partial}{\partial t} \int_{-\infty}^{t} i\, \lambda_a(t_o) \rho(a,t_o,t) dt_o$$

or

$$i \frac{\partial}{\partial t} \rho(a,t) = i\, \lambda_a(t)\, \rho_a + \int_{-\infty}^{t} \lambda_a(t_o)\, i \frac{\partial}{\partial t} \rho(a,t_o,t) dt_o$$

and using (2.17) we get

$$i \frac{\partial}{\partial t} \rho(a,t) = i\, \lambda_a\, \rho(a) + [H,\rho(a,t)] - \frac{i}{2} \{\Gamma,\rho(a,t)\} \qquad (3.7)$$

Similarly taking into account the possibilities that there are atoms being excited to the lower level 'b' and introducing the corresponding rate λ_b, the average pumping rate density we define for the two level system

$$\rho(\bar{r},t) = \rho(a,t) + \rho(b,t)$$

the equation of motion for $\rho(\bar{r},t)$ is in the two level system is given by

$$i\frac{\partial\rho}{\partial t} = i\lambda + [H,\rho] - \frac{i}{2}\{\Gamma,\rho\} \qquad (3.7a)$$

where λ is the pumping rate matrix given by

$$\lambda = \lambda_a\rho(a) + \lambda_b\,\rho(b) = \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix}$$

Here λ is the representation of 'atom pump reservoir' just as Γ represents the 'atom has reservoir'. The energy imput due to λ can be compensated for the energy loss due to Γ and leads to steady state solution, we may write

$$\begin{aligned} \rho(r,t) &= \rho(a,t) + \rho(b,t) \\ &= \int_{-\infty}^{t} \lambda_a(t_o)\rho(a,t_o,t)dt_o + \\ &\quad \int_{-\infty}^{t} \lambda_b(t_o)\rho(b,t_o,t)\,dt_o \\ &= \int_{-\infty}^{t} [\lambda_a(t_o)\rho(a,t_o,t) + \\ &\quad \lambda_b(t_o)\rho(b,t_o,t)]dt_o \end{aligned} \qquad (3.8)$$

From the definition of λ_a, λ_b and the form of the single atom operator Γ, the diagonal elements of $\rho(r,t)$ have the following physical interpretation.

$$\rho_{aa}(r,t) = N_a(r,t) = \text{population density of level a}$$

$$\rho_{bb}(r,t) = N_b(r,t) = \text{population density of level b}$$

And the density of population inversion is given by

$$\begin{aligned} \rho_{aa}(r,t) - \rho_{bb}(r,t) &= N_a(r,t) - N_b(r,t) \\ &= N(r,t) \end{aligned} \tag{3.9}$$

Using equation (2.16) we get

$$\begin{aligned} P(r,t) &= T_r[\rho(r,t)\,\hat{p}] \\ &= d[\rho_{ab}(r,t) + \rho^*_{ab}(r,t)] \end{aligned} \tag{3.9a}$$

Now we have complete set of equations of the self consisting theory which are given below.

The following equations determine the electric field $E(z,t)$

$$(\omega_n + \dot{\varphi}_n - \Omega_n) E_n = - \frac{\omega_n}{2\varepsilon_o} C_n$$

$$\dot{E}_n + \frac{\Omega}{2Q_n} E_n = - \frac{\omega_n}{2\varepsilon_o} S_n$$

$$E(z,t) = \sum A_n(t)\, u_n(z) = \sqrt{\tfrac{2}{L}} \sum A_n(t) \sin(k_n z)$$

$$A_n(t) = E_n(t) \cos(\omega_n t + \varphi_n(t))$$

$$P_n(t) = \int_0^L P(z,t)\, u_n(z)\, dz$$

$$P_n(t) = C_n(t) \cos[\omega_n t + \varphi_n(t)] + S_n(t) \sin[\omega_n t + \varphi_n(t)]$$

and the following equations determine how polarization depends on the electric field

$$i \frac{\partial \rho}{\partial t} = i\lambda + [H,\rho] - \frac{i}{2} \{\rho\, \Gamma, \rho\}$$

where

$$H = H_o + V = \begin{pmatrix} \omega_a & 0 \\ 0 & \omega_b \end{pmatrix} + \begin{pmatrix} 0 & V_{ab} \\ V_{ab} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \omega_a & V_{ab} \\ V_{ab} & \omega_b \end{pmatrix} \tag{3.10}$$

$$V_{ab} = \int \psi_a^* V \psi_b \, d\tau = V_{ba}^* = -E(r,t)\frac{d}{\hbar}$$

$$P(r,t) = d[\rho_{ab}(r,t) + \rho_{ab}^*(r,t)] = T_r[\rho(r,t)\hat{p}]$$

Now this dependence of polarization on electric field $E(z,t)$ is non-linear. The problem is to find a simultaneous solution of these two sets of equations corresponding to steady state solution. We may write equation (3.7a) in the following matrix form.

$$i\begin{pmatrix}\dot\rho_{aa} & \dot\rho_{ab}\\ \dot\rho_{ba} & \dot\rho_{bb}\end{pmatrix} = i\begin{pmatrix}\lambda_a & 0\\ 0 & \lambda_b\end{pmatrix} + \begin{pmatrix}H_{aa} & H_{ab}\\ H_{ba} & H_{bb}\end{pmatrix}\begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix}$$

$$-\begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix}\begin{pmatrix}H_{aa} & H_{ab}\\ H_{ba} & H_{bb}\end{pmatrix}$$

$$-\frac{i}{2}\left[\begin{pmatrix}\gamma_a & 0\\ 0 & \gamma_b\end{pmatrix}\begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix} + \begin{pmatrix}\rho_{aa} & \rho_{ab}\\ \rho_{ba} & \rho_{bb}\end{pmatrix}\begin{pmatrix}\gamma_a & 0\\ 0 & \gamma_b\end{pmatrix}\right]$$

whence

$$H = \begin{pmatrix} H_{aa} & H_{ab} \\ H_{ba} & H_{bb} \end{pmatrix}; \quad \Gamma = \begin{pmatrix} \gamma_a & 0 \\ 0 & \gamma_b \end{pmatrix} \text{ and } \lambda = \begin{pmatrix} \lambda_a & 0 \\ 0 & \lambda_b \end{pmatrix}$$

Taking the off diagonal term of the above matrix equation

$$i\,\dot{\rho}_{ab} = H_{aa}\rho_{ab} + H_{ab}\rho_{bb} - \rho_{aa}H_{ab} - \rho_{ab}H_{bb}$$

$$- \frac{i}{2}\gamma_a\rho_{ab} - \frac{i}{2}\rho_{ab}\gamma_b$$

or

$$i\,\dot{\rho}_{ab} = H_{aa}\rho_{ab} - \rho_{ab}H_{bb} + H_{ab}\rho_{bb} - \rho_{aa}H_{ab}$$

$$- \frac{i}{2}\rho_{ab}(\gamma_a + \gamma_b)$$

substituting from (3.10) for H_{aa}, H_{bb} and H_{bb} we have

$$i\,\dot{\rho}_{ab} = \omega_a\rho_{ab} - \rho_{ab}\omega_b + V_{ab}\rho_{bb} - \rho_{aa}V_{ab}$$

$$- \frac{i}{2}(\gamma_a + \gamma_b)\,\rho_{ab}$$

or

$$i\,\dot{\rho}_{ab} = \rho_{ab}(\omega_a - \omega_b) + V_{ab}(\rho_{bb} - \rho_{aa}) - \frac{i}{2}\rho_{ab}(\gamma_a + \gamma_b)$$

we may write the above

$$i\,\dot{\rho}_{ab} = \rho_{ab}(\omega_a - \omega_b) + V_{ab}(\rho_{bb} - \rho_{aa}) - i\,\rho_{ab}\gamma_{ab}$$

where

$$\gamma_{ab} = \frac{\gamma_a + \gamma_b}{2}$$

Hence equation of motion is

$$i\,\dot{\rho}_{ab} = \rho_{ab}\,\omega_o + V_{ab}(\rho_{bb} - \rho_{aa}) - i\,\rho_{ab}\gamma_{ab}$$

or

$$\dot{\rho}_{ab} = -i\,\omega_o\rho_{ab} - \rho_{ab}\gamma_{ab} + i\,V_{ab}(\rho_{aa} - \rho_{bb}) \quad (3.11)$$

Similarly for diagonal element we get

$$\dot{\rho}_{aa} = \lambda_a - \gamma_a\,\rho_{aa} + i\,V_{ab}\,(\rho_{ab} - \rho^{*}_{ab})$$

and

$$\dot{\rho}_{bb} = \lambda_b - \gamma_b\rho_{bb} + i\,V_{ab}(\rho_{ab} - \rho^{*}_{ab})$$

Hence we have the following off-diagonal and diagonal equations of motion ofor the element ρ corresponding to (3.7a)

$$\dot{\rho}_{ab} = -i\,\omega_o\rho_{ab} - \gamma_{ab}\rho_{ab} + iV_{ab}(\rho_{aa} - \rho_{bb}) \quad (3.11)$$

$$\dot{\rho}_{aa} = \lambda_a - \gamma_a\rho_{aa} + i\,V_{ab}(\rho_{ab} - \rho^{*}_{ab}) \quad (3.12)$$

$$\dot{\rho}_{bb} = \lambda_b - \gamma_b\,\rho_{bb} + i\,V_{ab}(\rho_{ab} - \rho^{*}_{ab}) \quad (3.13)$$

CHAPTER 5

LINEAR THEORY

The equations of motion

$$\dot{\rho}_{ab} = - i\omega_o \rho_{ab} - \gamma_{ab} \rho_{ab} + i V_{ab}(\rho_{ab} - \rho_{bb}) \qquad (3.11)$$

$$\dot{\rho}_{aa} = \lambda_a - \gamma_a \rho_{aa} + i V_{ab} (\rho_{ab} - \rho^*_{ab}) \qquad (3.12)$$

$$\dot{\rho}_{bb} = \lambda_b - \gamma_b \rho_{bb} - i V_{ab} (\rho_{ab} - \rho^*_{ab}) \qquad (3.13)$$

can be solved by an iterative perturbation series in powers of interaction V_{ab}. The first order solution will be linear in $E(z,t)$ and the higher order perturbation series correspond to non-linear solution. But this method is applied in more complicated case where we consider the atomic motion and Doppler broadening is taken into account. We will present here a more accurate solution by a different iteration method.

We define the linear approximation as that in which N (population inversion density) is assumed to be given and to be time independent; although the exact theory will be that in which $N(z,t)$ must be derived from the simultaneous solution of the equation of motions above.

According to our linear approximation

$$N = N(z)$$

The higher (non-linear) approximations are then obtained by iteration, starting with the linear approximation. Substituting $\rho_{ab} - \rho_{bb} = N(r,t) = N(z)$ in equation (3.11) we get

$$\dot{\rho}_{ab} + i\,\omega_o\,\rho_{ab} + \gamma_{ab}\,\rho_{ab} = i\,V_{ab}(\rho_{aa} - \rho_{bb})$$

or

$$\dot{\rho}_{ab} + (i\omega_o + \gamma_{ab})\,\rho_{ab} = i\,V_{ab}\,N(z) \qquad (4.1)$$

R.H.S. of equation (4.1) is linear in perturbation. Let us assume to begin with that only one mode n is excited, then from (1.9a)

$$E(z,t) = A_n(t)u_n(z)$$

Now

$$V_{ab} = -\,E(z,t)\,\frac{d}{\hbar}$$

$$V_{ab} = -\frac{d}{\hbar}\,A_n(t)\,u_n(z)$$

Substituting for $A_n(t)$ from (1.6)

$$V_{ab} = -\frac{d}{\hbar}\,E_n(t)\,\cos(\omega_n t + \varphi_n(t)u_n(z) \qquad (4.1a)$$

Substituting this in (4.1) we get

$$\dot{\rho}_{ab} + i(\omega_o+\gamma_{ab})\rho_{ab} = - i \frac{d}{\hbar} E_n(t)u_n(z)\cos(\omega_n t+\varphi_n(t)N(z)$$

$$= \frac{id}{\hbar} E_n(t)u_n(z) \times$$

$$[\frac{\exp\{-i(\omega_n t+\varphi_n(t)+\exp i(\omega_n t+\varphi_n(t))}{2}]N(z)$$

$$= - \frac{id}{2\hbar} E_n(t)\ u_n(z)[\exp-i(\omega_n t+\varphi_n(t)) +$$

$$\exp i(\omega_n t+\varphi_n(t))]\ N(z) \qquad (4.2)$$

In the above the first exponential is close to resonance with the atomic transition ($\omega_n \approx \omega_o$) and the second exponential is anti-resonant. Hence we may neglect the second exponential implyimg the rotating wave approximation.

Now setting

$$\rho_{ab}(t) = \xi(t)\ \exp(-i\ \omega_n t)$$

where $\xi(t)$ is slowly varying

$$\dot{\rho}_{ab}(t) = - i\ \omega_n \xi(t)\ e^{-i\omega_n t}$$

$$= - i\ \omega_n \rho_{ab}(t)$$

Hence equation of motion (4.2) becomes

$$-i\omega_n\rho_{ab}+(i\omega_o+\gamma_{ab})\rho_{ab} = \frac{id}{2\hbar} E_n(t)u_n(z)N(z)\exp[-i(\omega_n t+\varphi_n)]$$

or

$$\rho_{ab}[\gamma_{ab}+i(\omega_o-\omega_n)] = \frac{id}{2\hbar} E_n(t)\ u_n(z)\exp[-i(\omega_n t+\varphi_n)]$$

$$\rho_{ab} = \frac{-id}{2\hbar}\ \frac{E_n(t)u_n(z)}{[\gamma_{ab}+i(\omega_o-\omega_n)]}\ N(z)\ \exp[-i(\omega_n t+\varphi_n)] \quad (4.3)$$

which contains a resonant denominator.

Substituting the above result in (3.9a) and (1.11a) given below

$$P(r,t) = d[\rho_{ab} + \rho^*_{ab}] \qquad (3.9a)$$

$$P_n(t) = \int_o^L P(z,t)\ u_n(z)\ dz \qquad (1.11a)$$

we get,

$$P_n(t) = \int_o^L d(\rho_{ab}+\rho^*_{ab})u_n(z)\ dz$$

$$= \int_o^L d\ \frac{-id}{2\hbar}\ [\ \frac{E_n(t)u_n(z)N(z)}{\gamma_{ab}+i(\omega_o-\omega_n)}\ \exp[-i(\omega_n t+\varphi_n)]u_n(z)dz + c.c]$$

(c.c stands for complex conjugate).

$$= \frac{d^2}{2\hbar} E_n(t) \left\{ - i \bar{N}_n \frac{\exp[-i(\omega_n t + \varphi_n)]}{\gamma_{ab} + i(\omega_o - \omega_n)} + c.c \right\} \quad (4.4)$$

where

$$\bar{N}_n = \int_o^L N(z,t)\, u_n^2(z)\, dz \quad (4.5)$$

which in general depends on t (since we have written N(z,t)) but in our linear theory it does not depend on time. We have

$$P_n(t) = C_n(t) \cos(\omega_n t + \varphi_n) + S_n(t) \sin(\omega_n t + \varphi_n)$$

and writing

$$\frac{1}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2} = \mathcal{L}(\omega_n - \omega_o) \quad (4.6)$$

we can rewrite equation (4.4) in the following manner writing the complex conjugate

Now

$$\frac{- i \bar{N}_n \exp[-i(\omega_n t + \varphi_n)]}{\gamma_{ab} + i(\omega_o - \omega_n)} + c.c$$

is found to be

$$= -2\bar{N}_n \frac{[(\omega_o - \omega_n)\cos(\omega_n t + \varphi_n) + \gamma_{ab} \sin(\omega_n t + \varphi_n)]}{\gamma_{ab}^2 + (\omega_o - \omega_n)^2}$$

Putting this value in (4.4) we get

$$P_n(t) = -\frac{d^2}{\hbar} \frac{E_n \bar{N}_n (\omega_o - \omega_n)\cos(\omega_n t + \varphi_n)}{\gamma_{ab}^2 + (\omega_o - \omega_n)^2}$$

$$- \frac{d^2}{\hbar} \frac{E_n \bar{N}_n \gamma_{ab} \sin(\omega_n t + \varphi_n)}{\gamma_{ab}^2 + (\omega_o - \omega_n)^2}$$

Comparing the above with equation (1.17)

$$C_n(t) \cos(\omega_n t + \varphi_n) + S_n(t) \sin(\omega_n t + \varphi_n)$$

$$= -\frac{d^2}{\hbar} E_n \bar{N}_n (\omega_o - \omega_n) \cos(\omega_n t + \varphi_n) / \gamma_{ab}^2 + (\omega_o - \omega_n)^2$$

$$- \frac{d^2}{\hbar} E_n \bar{N}_n \gamma_{ab} \sin(\omega_n t + \varphi_n) / \gamma_{ab}^2 + (\omega_b - \omega_n)^2$$

equating the coefficients of cos and sin, we get

$$C_n(t) = \frac{d^2}{\hbar} \frac{E_n \bar{N}_n (\omega_n - \omega_o)}{\gamma_{ab}^2 + (\omega_n - \omega_o)^2}$$

and

$$S_n(t) = -\frac{d^2}{\hbar} \frac{E_n \bar{N}_n \gamma_{ab}}{\gamma_{ab}^2 + (\omega_n - \omega_o)^2}$$

Using (4.6) we have

$$C_n(t) = \frac{d^2}{\hbar} E_n \bar{N}_n (\omega_n - \omega_o) \mathcal{L}(\omega_n - \omega_o) \quad (4.7)$$

$$S_n(t) = -\frac{d^2}{\hbar} E_n \bar{N}_n \gamma_{ab} \mathcal{L}(\omega_n - \omega_o) \quad (4.8)$$

The above equation are linear in E_n. From (1.21)

$$\dot{E}_n + \frac{\Omega_n}{2Q_n} E_n = - \frac{\omega_n}{2\varepsilon_o} S_n$$

We have seen that for laser oscillation

$$\omega_n \approx \Omega_n$$

Therefore

$$\dot{E}_n + \frac{\omega_n}{2Q_n} E_n = - \frac{\omega_n}{2\varepsilon_o} S_n$$

Putting the value of S_n from (4.8) in the above relation, we get

$$\dot{E}_n + \frac{\omega_n}{2Q_n} E_n = \frac{d}{\hbar} \frac{\omega_n}{2\varepsilon_o} E_n \bar{N}_n \gamma_{ab} \mathcal{L}(\omega_n - \omega_o)$$

or

$$\frac{\dot{E}_n}{E_n} = - \frac{\omega_n}{2Q_n} + \frac{\omega_n}{2\varepsilon_o} \frac{d^2}{\hbar} \bar{N}_n \gamma_{ab} \mathcal{L}(\omega_n - \omega_o)$$

or

$$\frac{\dot{E}_n}{E_n} = - \frac{\omega_n}{2} \left[\frac{1}{Q_n} - \frac{d^2}{\varepsilon_o \hbar} \bar{N}_n \gamma_{ab} \mathcal{L}(\omega_n - \omega_o) \right] \qquad (4.9)$$

At the saturation value of field $\dot{E}_n = 0$ and putting this condition in the equation (4.9) we get the condition for laser oscillation at frequency ω_n.

$$-\frac{\omega_n}{2}\left[\frac{1}{Q_n} - \frac{d^2}{\varepsilon_o\hbar}\bar{N}_n\,\gamma_{ab}\,\mathcal{L}(\omega_n - \omega_n)\right] \leqslant 0$$

or

$$\frac{1}{Q_n} \leqslant \frac{d^2}{\varepsilon_o\hbar}\bar{N}_n\,\gamma_{ab}\,\mathcal{L}(\omega_n - \omega_o)$$

or

$$\frac{d^2}{\varepsilon_o\hbar}\bar{N}_n\,\gamma_{ab}\,\mathcal{L}(\omega_n - \omega_o) \geqslant \frac{1}{Q_n} \qquad (4.10)$$

The critical density of population inversion $\bar{N}_{nc}$ is obtained by evaluating equation (4.10) at resonance $\omega_n = \omega_o$.

$$\text{at} \quad \omega_n = \omega_o; \quad \mathcal{L}(\omega_n - \omega_o) = \frac{1}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2} = \frac{1}{\gamma_{ab}^2}$$

Hence

$$\frac{d^2}{\varepsilon_o\hbar}\bar{N}_{nc}\,\frac{\gamma_{ab}}{\gamma_{ab}^2} = \frac{1}{Q_n}$$

or

$$\bar{N}_{nc} = \frac{\varepsilon_o\hbar\,\gamma_{ab}}{d^2 Q_n} \qquad (4.11)$$

This is the expression for critical density of population inversion.

The frequency equation is

$$(\omega_n + \dot{\varphi}_n - \Omega_n)E_n = -\frac{\omega_n}{2\varepsilon_o}C_n$$

Substituting for C_n from (4.7) in the above equation and neglecting $\dot{\varphi}_n E_n$, we get

$$(\omega_n - \Omega_n) E_n = - \frac{\omega_n}{2\varepsilon_o} \frac{d^2}{\hbar} \bar{N}_n (\omega_n - \omega_o) \mathcal{L}(\omega_n - \omega_n) E_n \tag{4.12}$$

$\bar{N}_n$ corresponding to threshold of oscillation at ω_n is obtained from (4.10)

$$\bar{N}_n = \frac{1}{Q_n} \frac{\varepsilon_o \hbar}{\gamma_{ab} \mathcal{L}(\omega_n - \omega_o) d^2}$$

and putting this value of $\bar{N}_n$ in (4.12) we get

$$(\omega_n - \Omega_n) = - \frac{\omega_n}{2\varepsilon_o} \frac{d^2}{\hbar} \frac{\varepsilon_o \hbar}{Q_n \gamma_{ab}} \frac{(\omega_n - \omega_o) \mathcal{L}(\omega_n - \omega_o)}{\mathcal{L}(\omega_n - \omega_o) d^2}$$

or

$$\omega_n - \Omega_n = - \frac{\omega_n}{2Q_n} \frac{(\omega_n - \omega_o)}{\gamma_{ab}}$$

Writing

$$S = \frac{\omega_n}{2Q_n \gamma_{ab}}$$

We get

$$\omega_n - \Omega_n = - S(\omega_n - \omega_o) \tag{4.13}$$

Equating (4.13) represents the linear pulling effect. From (4.13)

$$\omega_n - \Omega_n = -S\omega_n + S\omega_o$$

or

$$\omega_n + S\omega_n = S\omega_o + \Omega_n$$

$$\omega_n = \frac{\Omega_n + S\omega_o}{1+S} \qquad (4.14)$$

The above equation (4.14) means that the frequency of laser oscillation in mode n is the weighted average of the mode frequency Ω_n and the atomic transition frequency ω_o with weights 1 and S respectively. For a gas laser the atomic line width is greater than the mode width. Hence $S \leqslant 1$ (Since $S = \frac{\text{mode width}}{\text{atomic line width}}$) for a gas laser, typical value of which is of the order 10^{-1} to 10^{-2}. Thus from (4.14) $\omega_n \simeq \Omega$ i.e. laser oscillation frequency is very close to mode frequency. If we regard the mode and atom as coupled oscillater, we see that the oscillator with smaller line width (large Q) pulls the frequency of oscillation close to its own. This is known as 'Frequency Pulling'.

If M modes are excited at laser frequency ω_n where $n = 1,2,\ldots,N$; we must replace (4.1a) by

$$V_{ab} = -\frac{d}{\hbar} \sum_{n=1}^{M} E_n(t)\, u_n(z) \cos(\omega_n t + \varphi_n) \qquad (4.15)$$

In linear rotating wave approximation

$$\rho_{ab} = -\frac{id}{\hbar} \sum_{n=1}^{M} \frac{E_n(t)u_n(z)N(z)}{\gamma_{ab}+i(\omega_o-\omega_n)} \times$$

$$\exp[-i(\omega_n t + \varphi_n)] \tag{4.16}$$

This is the solution in linear theory.

CHAPTER 6

NONLINEAR THEORY

(a) STEADY STATE SOLUTION OF THE ATOMIC EQUATION

In the linear approximation theory we had assumed

$$\rho_{aa} - \rho_{bb} = N(z)$$

as time independent and calculated corresponding value of ρ_{ab}. The next order approximation which will be nonlinear is obtained by assuming the population inversion density ($\rho_{aa} - \rho_{bb} = N(z,t)$) as slusly varying in time. Thus at any given time ρ_{ab} is still given by

$$\rho_{ab} = -\frac{id}{2\hbar} \sum_{n=1}^{M} \frac{E_n(t)\; u_n(z)}{\gamma_{ab} + i(\omega_o - \omega_n)} N(z,t) \times$$

$$\exp[-i(\omega_n t + \varphi_n)] \qquad (5.1)$$

Here N(z) of equation (4.16) is replaced by N(z,t). Thus

$$\rho_{ab} = -\frac{id}{2\hbar} (\rho_{aa} - \rho_{bb}) \sum_{n=1}^{M} \frac{E_n(t)\; u_n(z)}{\gamma_{ab} + i(\omega_o - \omega_n)} \times$$

$$\exp[-i(\omega_n t + \varphi_n)] \qquad (5.2)$$

substituting the above solution in the diagonal equations (3.12) and (3.13) one can solve for $\rho_{aa} - \rho_{bb}$.

From (4.1a)

$$V_{ab} = -\frac{d}{\hbar} \sum_{n=1}^{M} E_n(t)\, u_n(z) \cos(\omega_n t + \varphi_n)$$

From (4.1a) and (4.3) we get

$$V_{ab}\, \rho_{ab} = (\rho_{aa} - \rho_{bb}) \frac{id^2}{2\hbar^2} \sum_{m=1}^{M} E_m(t)\, u_m(z) \cos(\omega_m t + \varphi_n) \times$$

$$\sum_{n=1}^{M} \frac{E_n(t)\, u_n(z)}{\rho_{ab} + i(\omega_o - \omega_n)} \exp[-i(\omega_n t + \varphi_n)]$$

and

$$V_{ab}\rho^*_{ab} = -(\rho_{aa} - \rho_{bb}) \frac{id^2}{2\hbar^2} \sum_{m=1}^{M} E_m u_m(z) \cos(\omega_m t + \varphi_m) \times$$

$$\text{c.c of} \sum_{n=1}^{M} \frac{E_n(t)\, u_n(z)}{\gamma_{ab} + i(\omega_o - \omega_n)} \exp[-i(\omega_n t + \varphi_n)]$$

Thus

$$i(\rho_{ab} - \rho^*_{ab})V_{ab} = -\frac{d^2}{2\hbar^2} \sum_{m=1}^{M} E_m u_m(z) \cos(\omega_m t + \varphi_m)(\rho_{aa} - \rho_{bb}) \times$$

$$\left[\sum_{n=1}^{M} \frac{E_n u_n(z)}{\gamma_{ab} + i(\omega_o - \omega_n)} \exp[-i(\omega_n t + \varphi_n)] + \text{c.c} \right]$$

$$= -\frac{d^2}{4\hbar^2} \sum_{m=1}^{M} E_m u_m(z) [e^{-i(\omega_m t + \varphi_m)} + e^{-i(\omega_m t + \varphi_n)}](\rho_{aa} - \rho_{bb})$$

$$\sum_{n=1}^{M} \left\{ \frac{E_n u_n(z)}{\gamma_{ab}+i(\omega_o-\omega_n)} e^{-i(\omega_n t+\varphi_n)} + c.c \right\}$$

Applying rotating wave approximation, we get

$$i V_{ab}(\rho_{ab} - \rho^*_{ab}) = -\frac{d^2}{4\hbar^2} \sum_{m=1}^{M} \sum_{n=1}^{M} \left[\frac{E_m E_n u_m u_n}{\gamma_{ab}+i(\omega_o-\omega_n)} \times \right.$$

$$\left. e^{i\{(\omega_m-\omega_n)t+\varphi_m-\varphi_n\}} \right](\rho_{aa}-\rho_{bb} \qquad (5.3)$$

Writing

$$\frac{d^2}{4\hbar^2} \sum_{m=1}^{M} \sum_{n=1}^{M} \frac{E_m E_n u_m u_n}{\gamma_{ab}+i(\omega_o-\omega_n)} e^{i\{(\omega_m-\omega_n)t+\varphi_m-\varphi_n\}} + c.c = R$$

(5.4a)

Equation (5.3) becomes

$$iV_{ab}(\rho_{ab} - \rho^*_{ab}) = -R(\rho_{aa} - \rho_{bb}) \qquad (5.4)$$

Putting the above in the diagonal rate equations (3.12) and (3.13) written below

$$\dot{\rho}_{aa} = \lambda_a - \gamma_a \rho_{aa} + i V_{ab}(\rho_{ab} - \rho^*_{ab})$$

$$\dot{\rho}_{bb} = \lambda_b - \gamma_b \rho_{bb} - i V_{ab}(\rho_{ab} - \rho^*_{ab})$$

We get

$$\dot{\rho}_{aa} = \lambda_a - \gamma_a \rho_{aa} + R(\rho_{bb} - \rho_{aa}) \qquad (5.5)$$

$$\dot{\rho}_{bb} = \lambda_b - \gamma_b \rho_{bb} + R(\rho_{aa} - \rho_{bb}) \qquad (5.6)$$

These are the rate equations for the population densities ρ_{aa} and ρ_{bb} of level a and b. The first two terms represent the pumping and damping rates respectively associated with the atom reservoir. The last term represents the effects of induced emission and absorption which decreases the population of level a and increases the population of level b, when there is population inversion i.e. when $(\rho_{aa}-\rho_{bb}) > 0$ It will be shown in the following that $R \geqslant 0$ and that R can be interpreted as induced transition rate. It can be seen from (5.4a) that R is quadratic in the mode amplitude (E_m, E_n), hence it is proportional to intensity.

We observe from (5.4a) that if more than one mode is excited, R contains pulsating component corresponding to $(\omega_m - \omega_n)$. They will lead to the pulsation effect and the polarization would contain combination tones which is the characteristic of a nonlinear oscillator. For a typical gas laser $(\omega_m - \omega_n) \geqslant \gamma_{ab}$ (because the separation between the axial mode is of the order of 150 mc, whereas $\gamma_{ab} \sim 10$ mc); so the pulsating components are small and they will be neglected here. Thus $m = n$ can be substituted, because then pulsating component $(\exp\{\ \} = 1)$.

For $m = n$, we may write (5.4a) after putting for complex conjugate

$$R = \frac{d^2}{4\hbar^2} \sum_{n=1}^{M} \left[\frac{E_n E_n u_n u_n}{\gamma_{ab} + i(\omega_o - \omega_n)} + c.c \right]$$

$$= \frac{d^2}{4\hbar^2} \sum_{n=1}^{M} \frac{2(E_n^2 u_n^2)(\gamma_{ab})}{\gamma_{ab}^2 + (\omega_o - \omega_n)^2}$$

or

$$R \equiv \frac{d^2}{2\hbar^2} \sum_{n=1}^{M} \gamma_{ab}\, \mathcal{L}(\omega_n - \omega_o)\, E_n^2 u_n^2 \geqslant 0$$

The steady state solution of (5.5) and (5.6) is found by setting $\dot{\rho}_{aa} = \dot{\rho}_{bb} = 0$.

Thus

$$R(\rho_{bb} - \rho_{aa}) - \gamma_a \rho_{aa} + \lambda_a = 0$$

$$R(\rho_{aa} - \rho_{bb}) - \gamma_b \rho_{bb} + \lambda_b = 0$$

or,

$$\begin{aligned} \rho_{aa}(R+\gamma_a) - R\,\rho_{bb} &= \lambda_a \\ -\rho_{aa} R + \rho_{bb}(R+\gamma_b) &= \lambda \end{aligned} \qquad (5.7)$$

Solving the above simultaneous equation we get

$$\rho_{aa} = \frac{\lambda_a R + \lambda_a \gamma_b - R\lambda_b}{\gamma_a \gamma_b + R\gamma_a + R\gamma_b}$$

$$\rho_{bb} = \frac{R\lambda_b + \gamma_a \lambda_b - \lambda_a R}{\gamma_a \gamma_b + R\gamma_a + R\gamma_b}$$

Therefore

$$N = \rho_{aa} - \rho_{bb}$$

$$= \frac{\lambda_a R + \lambda_a \gamma_b - R\lambda_b - R\lambda_b - \gamma_a \gamma_b + R\lambda_a}{\gamma_a \gamma_b + R(\gamma_a + \gamma_b)}$$

$$= \frac{\lambda_a \gamma_b - \gamma_a \lambda_b + 2R(\lambda_a - \lambda_b)}{\gamma_a \gamma_b + R(\gamma_a + \gamma_b)}$$

assuming $\lambda_a \lambda_b = \lambda_{ab}$ we can write

$$N = \frac{\lambda_a \gamma_b + \gamma_a \lambda_b}{\gamma_a \gamma_b + R\gamma_{ab}}$$

Dividing the denominator and numerator by $\gamma_a \gamma_b$

$$N = \frac{N^{(0)}}{1 + \frac{R}{R_s}} \qquad (5.8)$$

where

$$N^{(0)} = \frac{\lambda_a}{\gamma_a} - \frac{\lambda_b}{\gamma_b}$$

and

$$\frac{1}{R_s} = \frac{1}{\gamma_a} + \frac{1}{\gamma_b} = \frac{\gamma_a + \gamma_b}{\gamma_a \gamma_b} = \frac{2\gamma_{ab}}{\gamma_a \gamma_b} = \tau_a + \tau_b$$

Thus in the nonlinear theory the population inversion density is given by (5.8). This relation explain good a number of phenomena of nonlinear theory of laser. In the absence of the field when $R = 0$ we get from (5.8)

$$N = N^{(0)}$$

Thus here $N^{(0)}$ represents zero field inversion density which is the consequence of equilibrium between the pumping and loss rates.

We may say that in the presence of the field, the inversion density is reduced by a factor $1 + \frac{R}{R_s}$. This is the effect of the induced emission i.e. the reduction increases with the intensity because R is proportional to the intensity and as R increases the reduction factor $(1 + \frac{R}{R_s})$ also increases. This is the nonlinear saturation effect which prevents the exponential growth of the intensity which is found in linear theory. This nonlinear saturation effect determines the stable amplitude of oscillation of laser.

We may calculate ρ_{ab} in this nonlinear approximation by substituting N(z,t) for $\rho_{aa} - \rho_{bb}$ in the equation (5.1).

Hence in the nonlinear approximation

$$\rho_{ab} = -\frac{id}{2\hbar} \frac{N^{(o)}}{(1 + \frac{R}{R_s})} \sum_{n=1}^{M} \frac{E_n u_n}{\gamma_{ab} + i(\omega_o - \omega_n)} e^{-i(\omega_n t + \varphi_n)} \tag{5.9}$$

This differs from the linear approximation (4.16) only by the substitution

$$N(z) \rightarrow \frac{N^{(o)}}{1 + \frac{R}{R_s}} .$$

(b) FIELD EQUATION

We have obtained the solution (5.8) and (5.9) of the atomic rate equations. In order to find corresponding values of C_n and S_n we must substitute (5.9) in the equation for polarization (3.9) and equation (1.11a). The values S_n and C_n in the nonlinear theory differ from that of the linear theory only due to difference in the value N_n in the two approximation. This calculation will help us in explaining some of the aspects of gas laser.

In the linear approximation we had written N_n as

$$\bar{N}_n = \int_0^1 N(z,t)\; u_n^2(z)\; dz$$

In the nonlinear approximation we must replace N(z,t) by

$\dfrac{N^{(0)}}{1 + \frac{R}{R_s}}$ and $\bar{N}_n$ by $\bar{N}'_n$ (say) . Hence

$$\bar{N}'_n = \int_0^L \frac{N^{(0)}}{1 + \frac{R}{R_s}}\; u_n^2(z)\; dz \qquad (5.10)$$

Let us consider the case in which only one mode is excited (say n) with sufficiently weak intensity i.e. R is well below the saturation. Hence

$$\frac{R}{R_s} \ll 1$$

Thus we may expand

$$\frac{1}{1 + \frac{R}{R_s}} = \left(1 + \frac{R}{R_s}\right)^{-1} = 1 - \frac{R}{R_s} + \ldots \qquad (5.11)$$

Higher order terms are neglected.

Putting (5.11) in equation (5.10) and taking the value of R and $\frac{1}{R_s}$, we get

$$\bar{N}'_n = \bar{N} - \frac{d^2 \gamma_{ab}^2}{\hbar^2 \gamma_a \gamma_b} \mathcal{L}(\omega_n - \omega_o) \int_o^L N^{(0)} u_n^4(z) E_n^2 \, dz \quad (5.12)$$

where

$$\bar{N} = \int_o^L N^{(0)} u_n^2(z) \, dz$$

and

$N^{(0)} = \frac{\lambda_a}{\gamma_a} - \frac{\lambda_b}{\gamma_b}$ is the pumping rate which is either constant or slowly varying that is it changes little with wave length.

We know that

$$u_n(z) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi z}{L} \right)$$

Then

$$\frac{1}{\bar{N}} \int_o^L N^{(0)} u_n^4(z) \, dz$$

$$= \frac{\int_o^L N^{(0)} u_n^4(z) \, dz}{\int_o^L N^{(0)} u_n^2(z) \, dz}$$

$$= \frac{\int_0^L u_n^4(z)\,dz}{\int_0^L u_n^2(z)\,dz}$$

$$= \frac{\frac{4}{L^2}\int_0^L \sin^4\left(\frac{n\pi z}{L}\right)dz}{\frac{2}{L}\int_0^L \sin^2\left(\frac{n\pi z}{L}\right)dz}$$

$$= \frac{2}{L}\,\frac{\int_0^L \sin^4\left(\frac{n\pi z}{L}\right)dz}{\int_0^L \sin^2\left(\frac{n\pi z}{L}\right)dz}$$

$$= \frac{3}{2L} \qquad (5.13)$$

From (5.12) we have

$$\bar{N}'_n = \bar{N} - \frac{d^2\gamma_{ab}^2}{\hbar^2}\,\mathcal{L}(\omega_n - \omega_o)\int_0^L N^{(0)}\,u_n^4(z)\,dz\;E_n^2$$

$$= \bar{N}\left[\,1 - \frac{d^2\gamma_{ab}^2}{\hbar^2\gamma_a\gamma_b}\,E_n^2\,\mathcal{L}(\omega_n - \omega_o)\,\frac{1}{\bar{N}}\int_0^L N^{(0)}u_n^4(z)\,dz\right]$$

Putting the value of the integral from (5.13) in the above equation we find

$$\bar{N}'_n = \bar{N} \left[1 - \frac{d^2\gamma_{ab}^2}{\hbar^2\gamma_a\gamma_b} E_n^2 \mathcal{L}(\omega_n - \omega_o) \frac{3}{2L}\right]$$

or

$$\bar{N}'_n = \bar{N} \left[1 - \gamma_{ab}^2 \mathcal{L}(\omega_n - \omega_o) I_n\right] \qquad (5.14)$$

where

$$I_n = \frac{3}{2L} \frac{d^2}{\hbar^2\gamma_a\gamma_b} E_n^2 \qquad (5.15)$$

In here is a measure of the intensity of the laser field of the mode n and it is a dimensionless parameter.

Within the approximation $\frac{1}{1 + \frac{R}{R_s}} = 1 - \frac{R}{R_s}$ we may write

$$1 - \gamma_{ab}^2 \mathcal{L}(\omega_n - \omega_o) I_n = \frac{1}{\left[1 - \gamma_{ab}^2 \mathcal{L}(\omega_n - \omega_o) I_n\right]^{-1}}$$

$$= \frac{1}{1 + \gamma_{ab}^2 \mathcal{L}(\omega_n - \omega_o) I_n}$$

Thus

$$\bar{N}'_n \approx \frac{\bar{N}}{1 + \gamma_{ab}^2 \mathcal{L}(\omega_n - \omega_o) I_n}$$

We know that

$$\mathcal{L}(\omega_n - \omega_o) = \frac{1}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2}$$

Therefore

$$\mathcal{L}^{-1}(\omega_n - \omega_o) = (\omega_n - \omega_o)^2 + \gamma_{ab}^2$$

Thus

$$\bar{N}'_n = \frac{\bar{N}\,\mathcal{L}^{-1}(\omega_n - \omega_o)}{(\omega_n - \omega_o)^2 + (1+I_n)\gamma_{ab}^2} \tag{5.16}$$

In the linear theory we had obtained

$$S_n = -\frac{d^2}{\hbar}\,\bar{N}_n\,\gamma_{ab}\,\mathcal{L}(\omega_n - \omega_o) \tag{4.8}$$

In the nonlinear theory $\bar{N}_n$ must be replaced by $\bar{N}'_n$ given by equation (5.16). Thus

$$S_n = -\frac{d^2}{\hbar}\,\frac{\bar{N}\,\mathcal{L}^{-1}(\omega_n - \omega_o)\,\mathcal{L}(\omega_n - \omega_o)\,E_n\gamma_{ab}}{(\omega_n - \omega_o)^2 + (1+I_n)\gamma_{ab}^2}$$

or

$$S_n = -\frac{d^2}{\hbar}\,\frac{N\,E_n\gamma_{ab}}{(\omega_n - \omega_o)^2 + (1+I_n)\gamma_{ab}^2} \tag{5.17}$$

equation (4.8) can be written as

$$S_n = -\frac{d^2}{\hbar}\,\frac{\bar{N}\,\gamma_{ab}\,E_n}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2}$$

Thus comparing (5.17) of the nonlinear theory with the equation (4.8) of the linear theory we see that the effective resonance width has increased by a factor $\sqrt{1+I_n}$ due to induced emission.

Substituting for S_n from (5.17) in the amplitude equation given below

$$\dot{E}_n + \frac{\Omega_n}{2Q_n} E_n = -\frac{\omega_n}{2\varepsilon_o} S_n$$

we get

$$\dot{E}_n + \frac{\Omega_n}{2Q_n} E_n = \frac{\omega_n}{2\varepsilon_o} \frac{d^2}{\hbar} \frac{\bar{N}\gamma_{ab} E_n}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2(1+I_n)}$$

or

$$\frac{\dot{E}_n}{E_n} + \frac{\Omega_n}{2Q_n} = \frac{\omega_n}{2\varepsilon_o} \frac{d^2}{\hbar} \frac{\bar{N}\gamma_{ab}}{(\omega_n - \omega_o)^2 + (1+I_n)\gamma_{ab}^2}$$

or

$$-\frac{\dot{E}_n}{E_n} = \frac{\Omega_n}{2Q_n} - \frac{\omega_n}{2\varepsilon_o} \frac{d^2}{\hbar} \frac{\bar{N}\gamma_{ab}}{(\omega_n - \omega_o)^2 + (1+I_n)\gamma_{ab}}$$

If we take $\omega_n \simeq \Omega_n$, then we get

$$-\frac{2}{\omega_n} \frac{\dot{E}_n}{E_n} = \frac{1}{Q_n} - \frac{d^2}{\varepsilon_o\hbar} \bar{N} \frac{\gamma_{ab}}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2(1+I_n)} \qquad (5.18)$$

This is a nonlinear differential equation and has a steady state solution $\dot{E}_n = 0$. We get E_n or I_n by setting R.H.S. equal to zero. Thus

$$\frac{1}{Q_n} - \frac{d^2}{\varepsilon_o \hbar} \bar{N} \frac{\gamma_{ab}}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2(1+I_n)} = 0$$

or

$$\frac{1}{Q_n} = \frac{d^2}{\varepsilon_o \hbar} \bar{N} \frac{\gamma_{ab}}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2(1+I_n)}$$

or

$$\frac{\varepsilon_o \hbar}{Q_n d^2 \bar{N}} = \frac{\gamma_{ab}}{(\omega_n - \omega_o)^2 + \gamma_{ab}^2(1+I_n)}$$

or

$$(\omega_n - \omega_o)^2 + \gamma_{ab}^2 (1+I_n) = \frac{\gamma_{ab} Q_n^2 d^2 \bar{N}}{\varepsilon_o \hbar}$$

or

$$(\omega_n - \omega_o)^2 + \gamma_{ab}^2(1+I_n) = \frac{\bar{N}}{\bar{N}_c} \gamma_{ab}^2$$

where

$$\bar{N}_c = \frac{\varepsilon_o \hbar}{d^2 Q_n} = \text{Critical inversion Density}$$

or

$$\gamma_{ab}^2(1+I_n) = \gamma_{ab}^2 \frac{\bar{N}}{\bar{N}_c} - (\omega_n - \omega_o)^2$$

or

$$\gamma_{ab}^2 \cdot I_n = \gamma_{ab}^2 \frac{\bar{N}}{\bar{N}_c} - (\omega_n - \omega_o)^2 - \gamma_{ab}^2$$

Therefore

$$I_n = \frac{\bar{N}}{\bar{N}_c} - 1 - \frac{(\omega_n - \omega_o)^2}{\gamma_{ab}^2} \tag{5.18}$$

In particular at resonance i.e. at $\omega_n \approx \omega_o$, equation (5.18) reduces to

$$I_n = \frac{\bar{N}}{\bar{N}_c} - 1 \tag{5.19}$$

That is the dimensionless intensity I_n of mode n is equal to the fractional excess of the inversion density over the critical inversion.

when $\bar{N} = \bar{N}_c$ we have $I_n = 0$

which is in agreement with the definition of the threshold.

$\frac{\bar{N}}{\bar{N}_c}$ is called the Relative Excitation

Equation (5.18) suggests that the intensity decreases outside the resonance by the square of the ratio of deturing $(\omega_n - \omega_o)$ to the width γ_{ab}.

CHAPTER 7

MODE COMPETITION

(a) MULTI-MODE

Uptil now we have discussed the laser oscillation in only one mode (say n) and found some interesting results. Now we will see what happens when there is possibility of excitation of more than one modes. We will use here also the same approximation $\frac{R}{R_s} \ll 1$.

Thus

$$\frac{1}{1+\frac{R}{R_s}} = 1 - \frac{R}{R_s} + \ldots$$

taking only first order term where

$$R = \frac{d^2}{2h^2} \sum_{n=1}^{M} \gamma_{ab}\, \mathcal{L}(\omega_n - \omega_o)\, E_n^2\, u_n^2(z)\, dz$$

and

$$R_s = \frac{2\gamma_{ab}}{\gamma_a \gamma_b}$$

We know that

$$\bar{N}'_n = \int_o^L \frac{N^{(0)}}{1+\frac{R}{R_s}}\, u_n^2\,(z)\, dz$$

$$= \int_0^L N^{(0)} \left(1 - \frac{R}{R_s}\right) u_n^2(z)\, dz$$

$$= \int_0^L N^{(0)} u_n^2(z)\, dz - N^{(0} \frac{R}{R_s} u_n^2(z)\, dz$$

$$= \bar{N} - N^{(0)} \frac{d^2}{2\hbar^2} \frac{1}{R_s} \int_0^L \sum_{n=1}^{M} \gamma_{ab}\, \mathcal{L}(\omega_n - \omega_o) \times$$

$$E_n^2\, u_n^2\, u_n^2\, dz$$

$$= \bar{N} - N^{(0)} \frac{d^2}{2\hbar^2} \frac{2\gamma_{ab}^2}{\gamma_a \gamma_b} \int_0^L \sum_{n=0}^{M} \mathcal{L}(\omega_n - \omega_o) u_n^2 u_n^2 E_n^2\, dz$$

$$= \bar{N} - \left[\frac{N^{(0)} d^2 \gamma_{ab}^2}{\hbar^2 \gamma_a \gamma_b} \int_0^L \sum_{n=1}^{M} \mathcal{L}(\omega_n - \omega_o) u_n^4\, E_n^2\, dz + \right.$$

$$\left. \int_0^L N^{(0)} \frac{d^2}{\hbar^2} \frac{\gamma_{ab}^2}{\gamma_a \gamma_b} \sum_{m \neq n}^{M} \mathcal{L}(\omega_m - \omega_o) E_m^2 u_m^2 u_n^2\, dz \right]$$

The first two terms are written with the help of equation (5.14) and (5.15) and then rewritting the above

$$\bar{N}_n' = \bar{N} - \gamma_{ab}^2 \frac{3}{2L} \frac{d^2}{\hbar^2 \gamma_a \gamma_b} E_n^2\, \mathcal{L}(\omega_n - \omega_o)\, \bar{N} -$$

$$\int_0^L N^{(0)} \frac{d^2}{\hbar^2} \frac{\gamma_{ab}^2}{\gamma_a \gamma_b} \sum_{m \neq n}^{M} \mathcal{L}(\omega_m - \omega_o) u_m^2\, u_n^2\, E_m^2\, dz$$

$$= \bar{N} - \frac{3}{2L} \frac{d^2\gamma_{ab}^2}{\hbar^2\gamma_a\gamma_b} \mathcal{L}(\omega_n - \omega_o) \bar{N} E_n^2 -$$

$$\frac{d^2}{\hbar^2} \frac{\gamma_{ab}^2}{\gamma_a\gamma_b} \sum_{m \neq n}^{M} \mathcal{L}(\omega_m - \omega_o) \int_o^L N^{(0)} u_m^2 u_n^2 E_m^2 dz \tag{6.1a}$$

We know that

$$u_m = \sqrt{\frac{2}{L}} \sin (K_m z)$$

$$u_n = \sqrt{\frac{2}{L}} \sin (K_n z)$$

Therefore the integral of the last term may be written

$$\int_o^L N^{(0)} \frac{2}{L} \sin^2(K_m z) \frac{2}{L} \sin^2(K_m z) dz$$

$$= \frac{1}{L^2} \int_o^L N^{(0)} [1 - \cos(2K_m z) - \cos(2K_n z) +$$

$$\frac{1}{2} \cos 2(K_m + K_n) z + \frac{1}{2} \cos 2(K_m - K_n) z] dz \tag{6.1}$$

If $N^{(0)}$ is either constant or slowly varying, then only the first and last terms within brackets given an appreciable contribution.

Let us now define

$$N_2(m-n) = \frac{1}{L} \int_0^L N^{(0)} \cos 2(K_m - K_n)\, z\, dz \qquad (6.2)$$

and we have seen that for $m = n$

$$\bar{N} \equiv \int_0^L N^{(0)} u_n^2(z)\, dz$$

$$= \frac{2}{L} \int_0^L N^{(0)} \sin^2(K_m z)\, dz$$

$$= \frac{1}{L} \int_0^L N^{(0)} (1 - \cos 2 K_n z)\, dz \qquad (6.3)$$

$$= \frac{1}{L} \int_0^L N^{(0)} dz - \frac{1}{L} \int_0^L N^{(0)} \cos (2K_n z)\, dz$$

$$= \frac{1}{L} \int_0^L N^{(0)} dz$$

Since second integral is zero.

Thus at $m = n$

$$N \equiv \frac{1}{L} \int_0^L N^{(0)} dz = N_o \quad (\text{say})$$

Neglecting the first and last term of equation (6.1) we get

$$\int_0^L N^{(0)}\, u_m^2\, u_n^2\, dz = \int_0^L \frac{1}{L^2} N^{(0)} \left[(1 - \cos 2K_m z) + \frac{1}{2}\cos 2(K_m - K_n) z\right] dz$$

$$= \frac{1}{L}\left[\frac{1}{L}\int_0^L N^{(0)} (1 - \cos 2K_m z)\, dz + \frac{1}{2}\,\frac{1}{L}\int_0^L N^{(0)} \cos 2(K_m - K_n) z\, dz\right.$$

Using (6.2) and (6.3) we get

$$\int_0^L N^{(0)}\, u_m^2\, u_n^2\, dz = \frac{1}{L}\left[\bar{N} + \frac{1}{2} N_{2(m-n)}\right] \qquad (6.4)$$

Putting (6.4) in (6.1a) we get

$$N'_n = \bar{N} - \frac{3}{2L}\,\frac{d^2}{h^2}\,\frac{\gamma_{ab}^2}{\gamma_a \gamma_b}\,\mathcal{L}(\omega_n - \omega_o)\,\bar{N}\,E_n^2 - \frac{d^2}{\hbar^2}\,\frac{\gamma_{ab}^2}{\gamma_a \gamma_b}\sum_{m \neq n}^{M} \mathcal{L}(\omega_m - \omega_o)\,\frac{1}{L}\left(\bar{N} + \frac{1}{2} N_{2(m-n)}\right) E_m^2$$

We have shown in the linear approximation that

$$S_n = -\frac{d^2}{\hbar}\,\bar{N}_n\,\gamma_{ab}\,\mathcal{L}(\omega_n - \omega_o)\,E_n$$

Now in our nonlinear theory $\bar{N}_n$ of the above equation must be replaced by $\bar{N}'_n$.

Thus

$$S_n = -\frac{d^2}{\hbar} \bar{N}'_n \gamma_{ab} \mathcal{L}(\omega_n - \omega_o) E_n$$

$$= -\frac{d^2}{\hbar} \gamma_{ab} \mathcal{L}(\omega_n - \omega_o) E_n [\bar{N} - \frac{3}{2L} \frac{d^2}{\hbar^2} \frac{\gamma_{ab}^2}{\gamma_a \gamma_b} \times$$

$$\mathcal{L}(\omega_n - \omega_o) \bar{N} E_n^2 -$$

$$\frac{1}{L} \frac{d^2}{\hbar^2} \frac{\gamma_{ab}^2}{\gamma_a \gamma_b} \sum_{m \neq n}^{M} \mathcal{L}(\omega_m - \omega_o)(\bar{N} + \frac{1}{2} N_{2(m-n)})]$$

Substituting this value of S_n in the amplitude equation

$$\dot{E}_n = -\frac{\omega_n}{2\varepsilon_o} S_n - \frac{\Omega_n}{2Q_n} E_n$$

We get

$$\dot{E}_n = -\frac{\omega_n}{2\varepsilon_o} [-\frac{d^2}{\hbar} \gamma_{ab} \mathcal{L}(\omega_n - \omega_o) E_n \bar{N} - \frac{d^2}{\hbar} \gamma_{ab} \mathcal{L}(\omega_n - \omega_o) \frac{3}{2L} \times$$

$$\frac{d^2}{\hbar^2} \frac{\gamma_{ab}^2}{\gamma_a \gamma_b} \mathcal{L}(\omega_n - \omega_o) \bar{N} E_n^2 E_n -$$

$$\frac{d^2}{\hbar} \gamma_{ab} \mathcal{L}(\omega_n - \omega_o) \frac{1}{L} \frac{d^2}{\hbar^2} \frac{\gamma_{ab}^2}{\gamma_a \gamma_b} \sum_{m \neq n}^{M} \mathcal{L}(\omega_m - \omega_o) \times$$

$$(\bar{N} + \frac{1}{2} N_{2(m-n)}) \; E_m^2 \, E_n] - \frac{\Omega_n}{2Q_n} E_n \tag{6.5}$$

Writing

$$\frac{\omega_n}{2\varepsilon_o} \frac{d^2}{\hbar} \bar{N} \gamma_{ab} \, \ell(\omega_n - \omega_o) \; E_n - \frac{\Omega_n}{2Q_n} E_n$$

$$= E_n [\frac{\omega_n}{2\varepsilon_o} \frac{d^2}{\hbar} \frac{\gamma_{ab}^2}{\gamma_{ab}} \bar{N} \, \ell(\omega_n - \omega_o) - \frac{\Omega_n}{2Q_n}]$$

$$= E_n \cdot \alpha_n$$

and writing

$$[- \frac{\omega_n}{2\varepsilon_o \hbar^3} \frac{d^4}{\gamma_a \gamma_b} \gamma_{ab}^3 \, \ell^2(\omega_n - \omega_o) \frac{3}{2} \frac{\bar{N}}{L}] \; E_n^3$$

$$= - \beta_n \cdot E_n^3$$

and

$$- \sum_{m \neq n}^{M} \frac{\omega_n}{2\varepsilon_o} \frac{d^4 \gamma_{ab}^3}{\hbar^3 \gamma_a \gamma_b} \, \ell(\omega_m - \omega_o) \, \ell(\omega_n - \omega_o) \times$$

$$\frac{1}{L} (\bar{N} + \frac{1}{2} N_{2(m-n)}) \; E_m^2 \, E_n$$

$$= - \sum_{m \neq n}^{M} \theta_{mn} \, E_m^2 \, E_n$$

We can express (6.5) in terms of α_n, β_n and θ_{mn} as

$$\dot{E}_n = \alpha_n E_n - \beta_n E_n^3 - \sum_{m \neq n}^{M} \theta_{mn} E_n E_m^2 \tag{6.6}$$

where $n = 1,2,\ldots,M$

where

$$\alpha_n = \frac{\omega_n d^2}{2\varepsilon_o \hbar} \frac{\bar{N}}{\gamma_{ab}} \gamma_{ab}^2 \; \mathcal{L}(\omega_n - \omega_o) - \frac{\Omega_n}{2Q_n}$$

$$\beta_n = \frac{\omega_n}{2\varepsilon_o} \frac{d^4 \gamma_{ab}^3}{\hbar^3 \gamma_a \gamma_b} \mathcal{L}^2(\omega_n - \omega_o) \frac{3}{2} \frac{\bar{N}}{L}$$

$$\theta_{mn} = \frac{\omega_n}{2\varepsilon_o} \frac{d^4 \gamma_{ab}^3}{\hbar^3 \gamma_a \gamma_b} \mathcal{L}(\omega_m - \omega_o) \mathcal{L}(\omega_n - \omega_o) \frac{1}{L} (\bar{N} + \frac{1}{2} N_{2(m-n)})$$

$$\approx \theta_{nm}$$

If we compare the value of α_n with reference to (4.9) we see that it represents the overall gain. Hence the condition of laser oscillation in mode n is $\alpha_n \geqslant 0$. The first term in α_n represents the gain due to balance between the atom pump and loss reservoir and the amplifying effect of induced emission. The second term of α_n represents the lass due to field loss reservoir. Thus we may say

α_n = gain (Atom reservoir + Induced Transition

– Loss (field Reservoir)

β_n is a parameter known as saturation parameter. We can explain it by considering a single mode n. Then (6.6) becomes

$$\dot{E}_n = \alpha_n E_n - \beta_n E_n^3 \tag{6.7}$$

E_n^3 corresponds to $E_n I_n$ of which represents a nonlinear saturation effect according to (5.12). From equation (6.7) we can find the intensity of single mode laser oscillation in the steady state ($\dot{E}_n = 0$)

$$\alpha_n E_n - \beta_n E_n^3 = 0$$

or

$$E_n(\alpha_n - \beta_n E_n^2) = 0$$

$$\alpha_n - \beta_n E_n^2 = 0$$

$$E_n^2 = \frac{\alpha_n}{\beta_n} = \frac{\text{gain} - \text{Loss}}{\text{Saturation parameter}} \tag{6.8}$$

The parameter θ_{mn} represents the nonlinear saturation effect on the coupling between different modes.

(b) TWO MODES

Now we shall discuss what happens when we consider the possibility of more than one cavity mode. We study the following nonlinear coupled differential equation (6.8)

$$\dot{E}_n = \alpha_n E_n - \beta_n E_n^3 - \sum_{m \neq n}^{M} \theta_{mn} E_n E_m^2$$

where $n = 1, 2, \ldots, M$

We take the case when only two adjacent modes (1,2) above threshold are excited. Equation (6.6) then becomes

$$\begin{aligned} \dot{E}_1 &= \alpha_1 E_1 - \beta_1 E_1^3 - \theta E_1 E_2^2 \\ \dot{E}_2 &= \alpha_2 E_2 - \beta_2 E_2^3 - \theta E_2 E_1^2 \end{aligned} \quad (6.9)$$

where $\theta = \theta_{12} \approx \theta_{21} > 0$, $\alpha_1 > 0$ and $\beta_1 > 0$

Let us introduce the intensity parameters

$$X = E_1^2 \text{ and } Y = E_2^2 \quad (6.9a)$$

substituting from (6.9a) in equation (6.9) we get

$$\begin{aligned} \dot{X} &= 2X(\alpha_1 - \beta_1 X - \theta Y) \\ \dot{Y} &= 2Y(\alpha_2 - \theta X - \beta_2 Y) \end{aligned} \quad (6.10)$$

Let us find the steady state solution of equation (6.10) i.e. when $X = 0$, $Y = 0$

Thus from equation (6.10) we obtain

$$0 = 2X(\alpha_1 - \beta_1 X - \theta Y)$$
$$0 = 2Y(\alpha_2 - \theta X - \beta_2 Y) \qquad (6.11)$$

According to (6.9a) only solutions in the first quadrant ($x \geqslant 0$, $y \geqslant 0$) make senses. From (6.11) we see that only four types of stationary solutions are possible

(1) $Y = 0;\quad \alpha_1 - \beta_1 X = 0 \quad$ i.e. $X = \dfrac{\alpha_1}{\beta_1}$

(2) $X = 0;\quad \alpha_2 - \beta_2 Y = 0 \quad$ i.e. $Y = \dfrac{\alpha_2}{\beta_2}$

(3) $X = Y = 0$

(4) In the steady state simultaneous oscillations in both the modes are achieved when

$$\alpha_1 - \beta_1 X - \theta Y = 0$$
$$\alpha_2 - \theta X - \beta_2 Y = 0 \qquad (6.12)$$

or

$$\beta_1 X + \theta Y = \alpha_1$$
$$\theta X + \beta_2 Y = \alpha_2 \qquad (6.13)$$

solution (1) represents laser oscillation in mode 1 above and solution (2) represents in mode 2 above. They represent the single mode operation in either mode with the other mode suppressed. Solution (3) is the trivial solution which is unstable above threshold. Solution will exist if the straight line L_1 and L_2 intersect in the first quadrant.

Let us now discuss the solution (4) in some detail. Solving (5.12) we get the steady state solution in the simultaneous two modes operation as

$$X = \frac{\alpha_1\beta_2 - \alpha_2\theta}{\beta_1\beta_2 - \theta^2} \qquad \text{where} \quad \beta_1\beta_2 \neq \theta^2 \qquad (6.15)$$

$$Y = \frac{\alpha_2\beta_1 - \alpha_1\theta}{\beta_1\beta_2 - \theta^2}$$

Let us investigate the stability of these modes. For this consider small derivation from the steady state solution for X and Y.

We write

$$X(t) = X + e_1(t)$$

$$Y(t) = Y + e_2(t)$$

Hence

$$X(t) = \dot{e}_1(t)$$

$$Y(t) = \dot{e}_2(t)$$

substituting for $\dot{X}$ and $\dot{Y}$ from above in equation (6.10) we get

$$\dot{e}_1 = 2(-X\beta_1 e_1 - \theta X e_2 + \beta_1 e_1^2 - \theta e_1 e_2)$$

$$\dot{e}_2 = 2(-\theta e_1 Y - \beta_2 e_2 Y - \theta e_1 e_2 - \beta_2 e_2^2)$$

Neglecting nonlinear terms in e's we get

$$\dot{e}_1 = 2(-X\beta_1 e_1 - \theta X e_2)$$

$$\dot{e}_2 = 2(-Y\theta e_1 - Y\beta_2 e_2)$$

or

$$\dot{e}_1 = -2X(\beta_1 e_1 + \theta e_2)$$

$$\dot{e}_2 = -2Y(\beta_2 e_2 + \theta e_1)$$

or

$$\begin{aligned} \dot{e}_1 &= -2X\beta_1 e_1 - 2X\theta e_2 \\ \dot{e}_2 &= -2Y\beta_2 e_2 - 2Y\theta e_1 \end{aligned} \qquad (6.16)$$

Writing in matrix form

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} 2X\beta_1 & 2X\theta \\ 2Y\theta & 2Y\beta_2 \end{pmatrix} \begin{pmatrix} -e_1 \\ -e_2 \end{pmatrix}$$

i.e.

$$\mathbf{Y} = \mathbf{A}\mathbf{X}$$

where

$$\mathbf{Y} = \begin{pmatrix} \dot{e}_1 \\ \\ \dot{e}_2 \end{pmatrix} ; \quad \mathbf{X} \equiv \begin{pmatrix} -e_1 \\ \\ -e_2 \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 2X\beta_1 & 2X\theta \\ \\ 2Y\theta & 2Y\beta_2 \end{pmatrix}$$

Now we solve the matrix equation for the eigen value λ.

Then

$$AX - \lambda X = 0$$

$$(\lambda I - A)X = 0$$

or

$$\lambda \begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -2X\beta_1 & 2X\theta \\ \\ 2Y\theta & 2Y\beta_2 \end{pmatrix} \begin{pmatrix} -e_1 \\ \\ -e_2 \end{pmatrix} = 0$$

or

$$\left[\begin{pmatrix} \lambda & 0 \\ \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2X\beta_1 & 2X\theta \\ \\ 2Y\theta & 2Y\beta_2 \end{pmatrix} \right] \begin{pmatrix} -e_1 \\ \\ -e_2 \end{pmatrix} = 0$$

or

$$\begin{pmatrix} \lambda - 2X\beta_1 & -2X\theta \\ \\ -2Y\theta & \lambda - 2Y\beta_2 \end{pmatrix} \begin{pmatrix} -e_1 \\ \\ -e_2 \end{pmatrix} = 0$$

The above system of homogeneous equations have non-trivial solution if and only if

$$\begin{vmatrix} \lambda - 2X\beta_1 & -2X\theta \\ -2Y\theta & \lambda - 2Y\beta_2 \end{vmatrix} = 0$$

which is the secular equation

$$\lambda^2 - 2(Y\beta_2 + X\beta_1)\lambda + 4XY\beta_1\beta_2 - 4XY\theta^2 = 0$$

The solutions are

$$\lambda = \frac{2(X\beta_1+Y\beta_2) \pm [4(X\beta_1+Y\beta_2)^2 - 4\cdot 4XY\beta_1\beta_2 + 4\cdot 4XY\theta^2]^{1/2}}{2}$$

i.e.

$$\lambda = (X\beta_1 + Y\beta_2) \pm [(X\beta_1 + Y\beta_2)^2 + 4XY\theta^2 - 4XY\beta_1\beta_2]^{1/2} \tag{6.19}$$

We have seen that steady state solution is obtained when $\beta_1\beta_2 \neq \theta^2$ when $\beta_1\beta_2 > \theta^2$ both roots λ_1 and λ_2 are positive in which case two mode solution is stable. This condition suggests that mode coupling is weak which is usually satisfied in a laser. When $\beta_1\beta_2 < \theta^2$ one of the roots is positive and the two modes solution is unstable. This is the case of strong coupling and the presence of one mode suppresses the other one and the solution is one or other of those in (1) and (2). The behaviour of two modes laser is illustrated in the Fig. (7.1). The figure is for the case $\alpha_1 = \alpha_2$ $2\beta_1 = \beta_2$. Both X,Y are non-zero as long as $\frac{\theta}{\beta} < 1$ i.e. $\beta > \theta$ (weak coupling) and when $\frac{\theta}{\beta} > 1$ i.e. $\theta > \beta$ one goes to $\frac{\alpha}{\beta}$ and the other goes to zero (strong coupling).

(c) PHASE PLANE ANALYSIS IN LINEARIZED APPROXIMATION

The behaviour of two modes laser oscillation can also be well understand qualitatively by studying the nature of the phase-plane trajectory of the coupled linear equation (6.16) which is the linearized approximation of the nonlinear coupled equations (6.10). The nature of the phase-plane trajectory will describe phenomenon of mode completition (i.e. which mode is suppressed and which mode is quenched and whether both modes exsist.

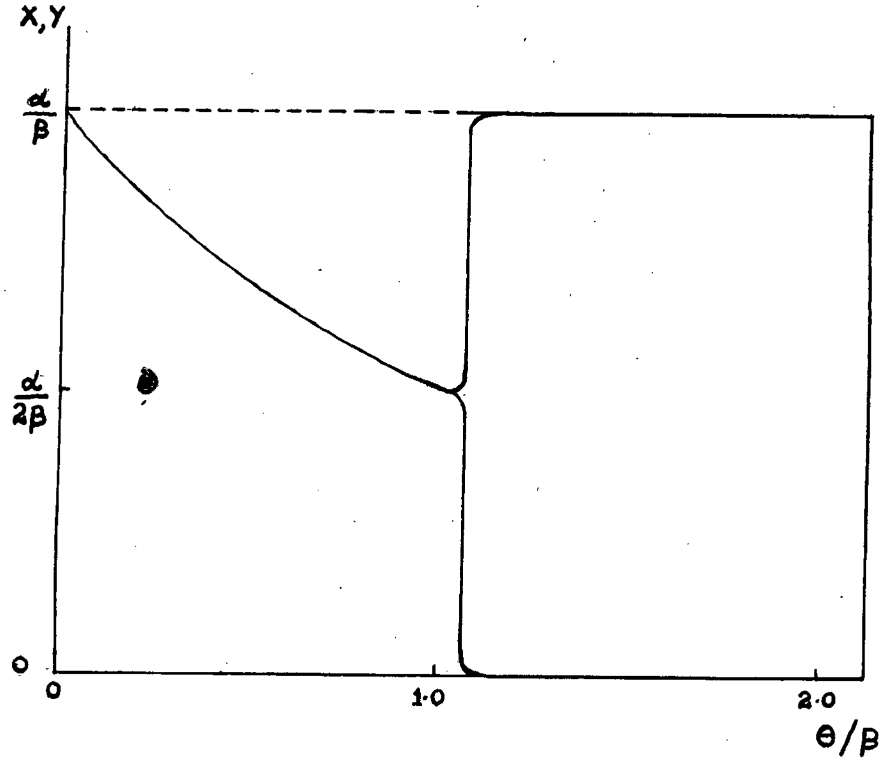
Our aim is to show that how far the above linearized approximation explains the nature of mode competition.

The linear coupled equations (6.10) are

$$\dot{e}_1 = -2X\beta_1 e_1 - 2X\theta e_2$$

$$\dot{e}_2 = -2Y\beta_2 e_2 - 2Y\theta e_1$$

where



FIG·7.1·The behaviour of the two intensities X,Y in two-mode operation·

$$X = \frac{\alpha_1\beta_1 - \alpha_2\theta}{\beta_1\beta_2 - \theta^2} \quad ; \quad Y = \frac{\alpha_2\beta_1 - \alpha_1\theta}{\beta_1\beta_2 - \theta^2}$$

We have solved the secular equation and found the characteristic roots (6.19) λ_1 and λ_2 in the previous section.

Therefore solution of (6.10) is

$$\begin{aligned} e_1 &= ae^{\lambda_1 t} + be^{\lambda_2 t} \\ \dot{e}_2 &= ce^{\lambda_1 t} + de^{\lambda_2 t} \end{aligned} \tag{6.20}$$

solving for $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ we get

$$e^{\lambda_1 t} = \frac{de_1 - be_2}{ad - bc}$$

and

$$e^{\lambda_2 t} = \frac{ae_2 - ce_1}{ad - bc}$$

we can write

$$e^{\lambda_1 t} = K(de_1 - be_2)$$

$$e^{\lambda_2 t} = K(ae_2 - ce_1)$$

where

$$K = \frac{1}{ad - bc}$$

Taking log of both sides of the above equation

$$\lambda_1 t = \log\,[K(de_1 - be_2)]$$

$$\lambda_2 t = \log\,[K(ae_2 - ce_1)]$$

$$\frac{\lambda_1}{\lambda_2} = \frac{\log[K(de_1 - be_2)]}{\log[K(ae_2 - ce_1)]}$$

or

$$\lambda_1 \log[K(ae_2 - ce_1)] = \lambda_2 \log[K(de_1 - be_2)]$$

or

$$\log[K(ae_2 - ce_1)]^{\lambda_1} = \log[K(de_1 - be_2)]^{\lambda_2}$$

or

$$[K(ae_2 - ce_1)]^{\lambda/\lambda_2} = K(de_1 - be_2)$$

or

$$(de_1 - be_2) = \frac{K^{\lambda_1/\lambda_2}}{K}\,(ae_2 - ce_1)^{\lambda_1/\lambda_2}$$

$$W = K'\,Z^{\lambda_1/\lambda_2} \qquad (6.21)$$

where

$$K' = K(\lambda_1/\lambda_2 - 1)$$

$$W = de_1 - be_2 \quad \text{and} \quad Z = ae_2 - ce_1$$

W and Z are the new variables corresponding to a rotation of e_1 and e_2 plane.

The nature of the phase plane trajectory depends upon the nature of the roots λ_1 and λ_2. We discuss below the different conditions in which we get different nature of the roots and consequently the behaviour of the trajectory which describes the mode competition.

Case 1

$\alpha_1 = 1$ (i.e. mode 1 well above threshold)

$\alpha_2 = 0.4$ (mode 2 having smaller gain)

$\beta_1 = \beta_2 = 2$ and $\theta = 1$.

Putting the above values in (6.19) we get the following two roots $\lambda_1 = 1.81$ and $\lambda_2 = -2.33$. Thus the two roots are real, unequal and opposite in sign. Therefore according to (6.21) the phase curve is hyperbolic (Fig.7.2) and we find that of the two eigen modes one suppresses the other as time increases. This is the case of only one mode oscillation. We can infer the same result from W - Z phase space diagram Fig. 7.2).

Case 2 When

$\alpha_1 = 1$ and $\alpha_2 = 1$ (i.e. both modes are well above threshold)

$\beta_1 = \beta_2 = 2$

$\theta = 1$

PHASE - PLANE TRAJECTORY

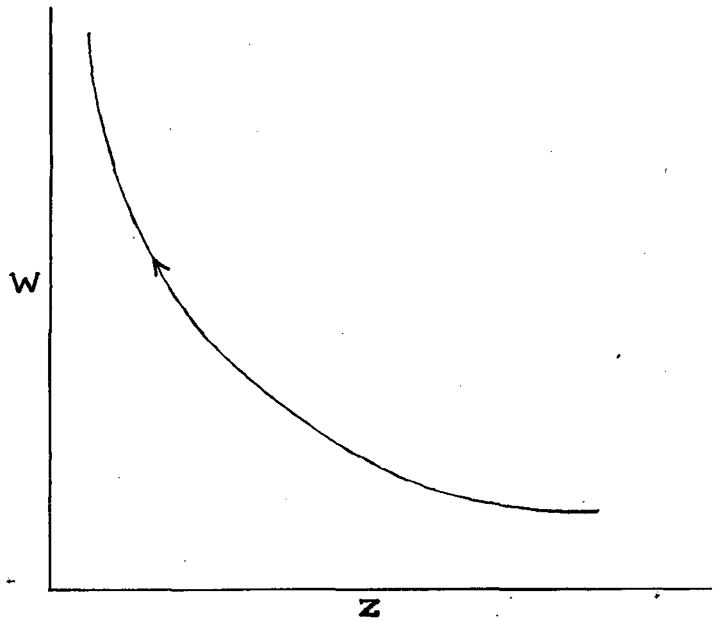


Figure I.2. λ_1, λ_2 real, unequal & opposite in sign.

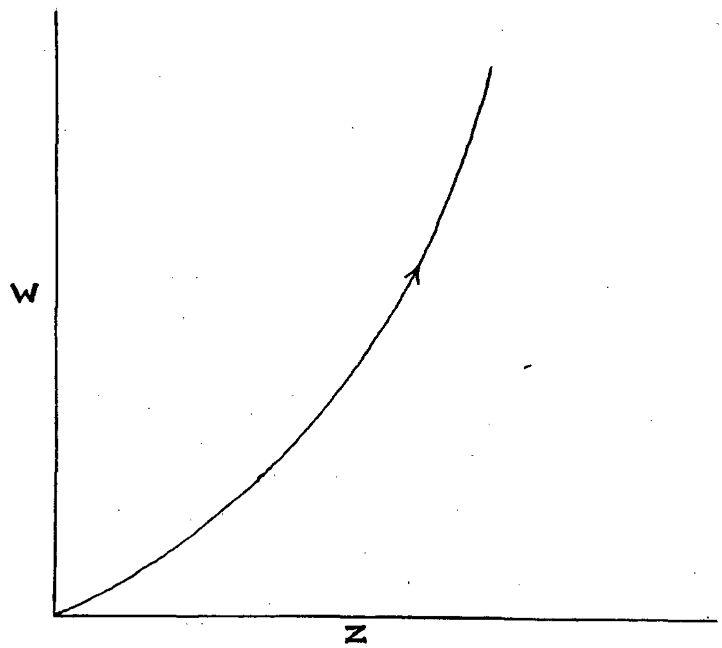


Figure I.3. λ_1, λ_2 real, unequal and positive.

We get $\lambda_1 = 2$ and $\lambda_2 = \frac{2}{3}$.

In this case the two roots are realy unequal and positive and thus the phase curve is a parabola (Fig. 6.2). Thus as time increases both the eigen modes grow. This is the case of simultaneous oscillation in both modes i.e. mode coupling is weak.

Case 3 When

$$\alpha_1 = \alpha_2 = 1$$

$$\beta_1 = \beta_2 = 1 \quad \text{and } \theta = 1$$

we get $\lambda_1 = 2$ and $\lambda_2 = -\frac{2}{3}$

The two roots are real, unequal and opposite in sign. Thus with increase of time one eigen mode decays and the other grows. That is only one mode oscillation is possible, which is the case of strong coupling. The phase space diagram is similar to Fig. (7.3).

It is interesting to note that the nature of mode competition which we obtained by the phase-plane analysis of linearized equations is well in agreement with the nature obtained by Lamb[1] by the direct numerical analysis of exact nonlinear equations.

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1. W.E. Lamb, Phy. Rev. 134 A 1429 (1964)

CHAPTER 9

CONCLUSION

In the semiclassical theory presented here we have treated the interaction of each atom separately with the field caused by all atoms. Our approach neglects both atom-atom correlation and atom-field correlations. To treat the atom independently is justified for large field because one atom can influence the amplitude very little. However, for a full treatment of laser one will have to take into account the quantum nature of the field.

The semiclassical theory presumably fails close to threshold as the assumption of atomic independence is invalid there. The laser oscillation are assumed to take place at a set of discrete frequencies w_n and consequently the line width of the laser is neglected. By including the classical noise (i) a finite line width is obtained. The line width obtained by the quantum theory is however twice that of obtained from classical theory. This difference is due to the spontaneous emission noise which has not been considered in the semiclassical theory.

(i) Lamb, W.E. Jr. Theory of optical maser, in Quantum optics and Electronics, Editors C.De Witt, A. Blendin and C. C. Cohen- Tannendji Gordan and B reach Science publication Inc. New York 1965.

This is the main deffects of the semi-classical theory.

Another consequence of the absence of spontaneous emission is that, in the semiclasical theory we have to assume that there already is a field present in the initial state, otherwise the excited atoms could not decay. In order to describe the growth of oscillations from an initial state when no radiation is present, we need quantum theory of laser.

The semiclassical theory considers the electromagnetic field classically and hence question of photon distribution does not arise. Recently experimental work in the field of photon counting has provided ample justification for considering the quantum nature of the field. This has been done recently by Wille's (1,2) Fleek.

We conclude that the semiclassical theory is able to explain most features of laser operation. It is invalid very close to threshold and questions that depends essentially on the quantum nature of the field, like the line width and photon statistics, demand a quantum-mechanical treatment. Thus the semiclassical treatment provides the most useful

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1. Wille's C.R., Quantum theory of Laser model, Phys. Rev. 147, 406 (1966).
2. Wille's C.R.; Quantum theory of a gas laser, Phys. Rev. 156, 320 (1967).

The first application of laser system in the field were reported, at nearly the same line by Fiaco and Smullin (1963), who recorded back scattered echoes from turbidity in the upper atmosphere using a pulsed ruby laser system, and by Ligda (1963) who measured back scatter from a molecular atmosphere and haze at low altitudes, and from clouds in the troposphere.

In addition to pulsed laser ranging system continuous wave (CW) lasers with or without some form of modulation have proven useful for maintaining environmental parameters. Such C.W. system have been used primarily for measures of atmospheric winds and turbulance.

With the laser only slightly more than 10 years old we are already have examples of semi-operational use of environmental monitoring, for cloud height detection and for urban pollution studies. As the requirement for environmental quality control increases in importance in the coming years, there will not doubt be increased emphasis in the operational use of lider (light detection and ranging) for monitoring pollution mixing depths, visibility, wind and for chemical analysis of pollutant constituents in the atmosphere. The ground work for such developments has been laid by the research and feasibility studies. There will certainly be additional applications which will prove visible as laser technology advances in the coming years.

-x-starting point for most applications of the laser theory.

APPLICATION OF LASERS IN ENVIRONMENTAL MONITORING

We have discussed some interesting theoretical features of lasers. In view of the growing problems of envirnment, it will not be impertinent to mention some of the application of laser in environmental monitoring.

Probing the environment with light is an old technique. Tyndall (1869) used an electric lamp to study the polarization of light scattered from smoke in his darkened nineteenth century laboratory. Hulburt (1937) studied atmospheric turbidity and molecular scattering to a height of 28 km in 1937 using a search light. Freidland et al (1956) used a pulsed searchlight technique for atmospheric probing and one of our present day turbid atmosphere model is based on such measurement by Elterman (1966,1968).

But it was the invention of the laser by Miaman (1960) and of the giant pulse technique by McClung and Hellwarth (1962) that revolutionized optical probing of the environment This potential was quickly realized by Goyer and Watson (1963) who considered the possibility of mapping the spatial and size distribution of droplets in cloudes and perhaps wind and turbulance by means of the Doppler frequency shift in back scattered light.