ON SOME PARADOXES IN THEORY OF VOTING: REVISITING THE ISSUE OF REPRESENTATIVENESS

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CERTIFICATE

This is to certify that the dissertation entitled **"On Some Paradoxes In Theory Of Voting: Revisiting The Issue Of Representativeness"** submitted by me in partial fulfillment of the requirement for the award of **MASTER OF PHILOSOPHY** has not been previously submitted for any other degree of this or any other university.

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We recommend that this dissertation be placed before the examiners for the evaluation.

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CONTENTS

I. Introduction

The theory of social choice deals explicitly with individual preferences within a society and their aggregation into social preferences or outcome, in other words a "social choice". According to Sen (1977) the realm of such inter-personal preference aggregation encompasses the following three different areas of problems; i) *Committee decisions* where a committee has to choose between alternative agendas or elect representatives on the relative merit of which every member holds different opinion. An obvious extension of this is where people of a nation decide their political destiny by voting according to their preferences; ii) *Social welfare judgements* which involves an individual's judgement about the society or some change in it which will bring about certain benefits for some section of people while depriving others; iii) *Normative indicators* involve measurement of indicators of general level of welfare of the society like measuring "national income", inequality and poverty or general level of social attainments like education, health, gender equality within a society. The theory of committee decisions is concerned with merely aggregating the views or opinion of members of a society or a committee on some given set of issues or agendas, the focus being primarily rested on the fairness of the procedure of arriving at a choice or whether a choice so made is optimal, i.e. represents the opinions of the members. On the other hand social welfare judgement concerns itself with aggregating personal welfare levels with the focus being on "optimality " of social welfare with frequent invoking of concepts of binary relations like "better" or "at least as good as" and interpersonal rankings of welfare levels.

Out of the above three strands of literature it is the theory of committee decisions or rather theory of voting that has historically occupied the centre stage. The earliest known contribution as documented by Mclean and Urken (1995) was writings by Pliny the Younger from second century AD, as well as Ramon Lull and his disciple Nicolas Cusanus from 13^{th} century AD. Pliny the Younger in his communiqué to Titius Aristo wrote about the proceedings of the senate where three very different sentences on a group of people were voted upon. The sentences being "death", "banishment" and "go free", Pliny argued that the best way to decide was by taking two sentences at a time and allowing a vote on them, the winner of the two being again pitched against the remaining sentence. Similarly both Ramon Lull and Nicolas Cusanus wrote about electoral procedures to be used to elect representatives of the church. The idea being electing a group of electors from the members by the method of plurality and this group of electors then going on to elect their representatives from amongst themselves by a *pair-wise comparison method* (Ramon Lull) similar to one suggested by Pliny or a point scoring method (Nicolas Cusanus). However such discussions in bits and pieces never transcended the barriers of philosophical or religious discourse until the great debate that took place between M. de Condorcet and M. de Borda, both members of the Paris Academy of Sciences, in the later part of 18^{th} century.

The discourse in the theory of social choice that deals with the theory of voting has been historically captivated with the idea of the fairness of the outcome of a process of preference aggregation. The earliest precursor to this line of thought which has shaped the development of modern social choice theory can be traced back to the well known mathematician of the late eighteenth century, Marquis de Condorcet. Condorcet was the first to use modern mathematical tools in the theory of voting to establish that any alternative that beats every other alternative by a majority in pair-wise comparison is the best candidate to be selected as the representative of the society. Around this time another noted mathematician and physicist in France, Jean Charles de Borda came out with another interpretation of the best representative for the society. Borda suggested that if a score is assigned for the position of a candidate in the preference rankings of all the individual voters, sum of the scores over all the preference ranking, for all the candidates gives social ranking over all the candidates and the candidate having the highest score is declared the winner. In Borda's method if ranks are assigned to each alternative on the basis of the number of other alternatives that are ranked below it then the Borda score of a candidate is equal to the total number of favourable votes obtained by a candidate in pair-wise comparison, in other words the alternative which has the most number of votes in pair-wise comparison is the Borda winner. Both the above two formulations give precedence to the idea of a majority prevailing over the judgements and deliberations of the society.

It is during this era social choice theory started taking firm root with the introduction and development of mathematical tools that were earlier never used in the discourse. A classic case in point in this regard is Condorcet's use of probability theory in estimating the social preference ordering that is closest to the preference of the majority. Similarly Borda while contending the efficacy of the plurality voting system, put forward his method based on rank ordering of the contestants in the individual voters' preference ranking. However simple their framework and contention may be, these attempts heralded new beginnings which although did not trigger immediately, but gradually started taking shape in the body of works of C.L. Dodgson on proportional representation and majority rule, Thomas Hare and H. R. Droop both separately on apportioning electoral seats on the basis of proportion of votes and others, finally culminating into the celebrated Arrow's Theorem in the middle of the bygone century. The democratic ethos developed over the past centuries in Europe as well as the new world states like in the America and Australia, led to the development of some notion of fairness in the democratic process. Arrow's theorem showed that there was no such process which satisfies all these notions of fairness taken together. Intuitively Arrow's theorem should have put to rest the centuries old quest for that perfect democratic process which would deliver to the society higher social welfare, instead a frenzy of activity started to break the shackles of the theorem which in turn led to the development in modern social choice theory.

Arrow's impossibility theorem shows that four independent conditions (Unrestricted domain, Pareto criterion, Independence of irrelevant alternatives and Non-dictatorship), which individually considered are plausible for every preference aggregation rule to satisfy, however taken together there is no such rule which satisfies all the four in tandem. This opened up a plethora of other conditions which independently are desirable for every preference aggregation rule to satisfy but taken together may turn out to be mutually inconsistent. The failure of the social choice rules to abide by these conditions primarily either due to the nature of their construction or mutual inconsistencies of these properties gave rise to a host of paradoxes in the theory of voting.

The primary motivation behind the standard axiomatic analysis in the classical social choice theory starting from Condorcet to the works till date originate from its quest to find a social choice rule that best approximates the preferences of individuals in the society. However, this pursuit has more often than not sidestepped the issue of representativeness of an outcome in favour of fairness of the process. It should be noted that fairness of an outcome and representativeness are two distinct issues to be dealt with separately and in the latter scant attention has been paid in social choice theory. Grofman (1981) in this regard points out that when voters are homogenous one can solve the problem of fair representation by solving the problem of equal representation. When voters are distinct in terms of their interests and preferences and cannot be treated interchangeably, fair representation and equal representation diverge. While aiming for equal representation for every individual in the society may be an idealistic pursuit and more relevant in the context of societies practicing direct democracy, proportional representation systems is more feasible in the context of representative democracies especially when the number of individuals seeking representation is very large. Equal representation however, can serve as an important benchmark in analyzing and comparing different voting rules as regards their propensity towards majoritarianism or ensuring minority representation. The problem which forms the epicenter of the debate between majoritarianism and proportional representation system has been quite succinctly articulated in the words of Thomas Gilpin[1] "*Let us therefore examine the question, whether there can be a legislative assembly elected, so as to represent the respective interests of the community in its deliberations and to allow the control of the majority in its decisions to which it is entitled.*" The tone of these words although sounds reconciliatory, it effectively summarizes the bone of contention of the above mentioned debate as majoritarian principle works by subverting minority representation and exaggerating by the process majority's dominance, while proportional representation systems by way of fostering minority representation factionalizes the polity thereby subverting effective governance.

[1]Chamberlin and Courant (1983).

This work pans the two alternate paradigms of analyzing theory of voting by charting out the developments in axiomatic analysis of voting through a survey of the paradoxes that pervade it, subsequently taking up the issue of representativeness through the debate between majoritarian principle and proportional representation.

II. Paradoxes in Theory of Voting

The paradoxes in social choice theory can serve as useful pointers in its development by highlighting on the numerous desiderata that have evolved in the literature as a bulwark against these paradoxes. The restrictions thus imposed on the preferences of individuals and processes of preference aggregation serve as the basis of analyzing the various voting rules in regard to their success in adherence to these restrictions and thereby approximating the preferences of the society from the individual preferences as closely as possible. The following sections in this chapter lay down the developments in social choice theory using some well known paradoxes that have bothered social choice theorists past and present. As the paradoxes are explained, a parallel effort has been made to elucidate on the violations of axioms they involve and the inconsistency results involving these axioms that these paradoxes entail.

1. Borda's Paradox

Jean-Charles de Borda in his presentation to the French Royal Academy contended that the plurality procedure then widely used for elections was inefficient in representing the opinion of the voters and in turn proposed a procedure based on rank-order count which is now famous as Borda count. Grazia (1953) provides a translation of the memoir in which Borda draws attention towards the inefficacy of plurality rule in aggregating voter's preferences when there are more than two candidates for election involved. Borda by way of the following example (Table 1) involving three candidates A, B, C and twenty one voters shows that by plurality rule though A wins, however there is a plurality of vote against A and it may not be the best choice.

Table 1: Plurality Rule and Borda's Paradox

1 voter	7 voters	7 voters	6 voters
A	A	B	C
B	C	C	B
C	B	A	A

From Table 1 it can be easily seen that a pair-wise comparison among the candidates renders A being defeated by both B and C by a 13 – 8 margin and C defeating B by a similar margin. In more modern terminology then C is the Condorcet winner while A is the Condorcet loser. As an alternative to the plurality rule Borda suggested a method which supposedly uses the entire preference profile to determine a winner. According to Borda's method each alternative is given a points for each voter that assigns it to the last rank. For second last position the point being $a+b$ and for third last position $a+2b$ and so on, assuming both a and b to be non-negative constants. It should be noted here that Borda provides for equal weights to distances between any two consecutive rankings in the preference ordering of the voters. In other words if there are three candidates to be ranked then the difference in the points of first ranked candidate ($a+2b$) and second ranked candidate ($a+b$) is equal to the difference between second ranked candidate and that of the candidate ranked last with (a) points. The points given by the voters to each candidate are summed to obtain the Borda scores and the candidate with the highest Borda score is the Borda winner. From the profile given in Table 1, assuming a = 0 and b = 1, the Borda scores can be calculated as:

A: $8 \times 2 + 13 \times 0 = 16$.

B: $7 \times 2 + 7 \times 1 = 21$.

C: $6 \times 2 + 14 \times 1 = 26$.

Therefore C is the Borda winner, on the merit that C has been ranked high enough by most of the voters and therefore C's general acceptability among the voters being more than any other candidate. In the above solitary example that Borda provided in his memoir it is merely coincidental that C is both the Borda winner and the Condorcet winner, but this is not the case for every preference profile so it is difficult to infer whether Borda intended his method to select a Condorcet winner always. However, what Borda's method succeeds in doing is elimination of the Condorcet loser (whenever it exists) from being a contender for election which seemed to be his primary case against plurality rule.

2. Condorcet's Paradox

Around the same time that Borda's memoir was published, one of his contemporaries M. de Condorcet came out with a thesis (titled: Essai sur l'application de l'analyse a la probabilite des decisions rendues a la pluralite des voix) in 1785 which suggested that an alternative that defeats every other alternative by a simple majority is the socially optimal choice[2]. Condorcet's main argument was that if the object of voting is to arrive at a socially optimal choice then the majority alternative (if it exists) is statistically the best choice. The primary contention in the above assertion being that, though individuals judge imperfectly but majority of individuals is correct in majority of the cases; this means probability of judging correctly is always greater than half for majority of the individuals. Condorcet's basic theorem asserts that if there are *h* individuals in the majority and *k* individuals in the minority, then the probability that the majority decision is correct is given by

$$\Pi = \frac{v^{h-k}}{v^{h-k} + e^{h-k}}$$

where *v*, the probability that an individual is correct is assumed > 0.5, while $e = 1 - v$, i.e. error in judgement. However Condorcet soon found that his probabilistic formulation does not necessarily choose the Condorcet winner. One of Condorcet's own example illustrates this problem (Mclean and Urken, 1995).

Table 2: Condorcet's Paradox

13 voters	10 voters	13 voters	6 voters	18 voters
A	A	B	B	C
C	B	C	A	B
B	C	A	C	A

Here C is the Condorcet winner defeating both A and B in pair-wise comparison by a margin of thirty one to twenty-nine votes. B defeats A by a margin thirty-seven to twenty-three and is the Borda winner having the largest plurality in pair-wise

[2] See Young (1988), Condorcet's Theory of Voting.

comparisons. According to Condorcet's formulations, the probability that C is the best choice is the joint probability that "C is better than B" and "C is better than A" are both true. Therefore probability for C being the best choice is given by:

$$\Pi(C) = \frac{v^2}{v^2+e^2}\frac{v^2}{v^2+e^2} = \frac{v^4}{v^4+2v^2e^2+e^4} \tag{1}$$

Similarly for B the joint probability can be calculated as:

$$\Pi(B) = \frac{v^{14}}{v^{14}+e^{14}}\frac{e^2}{v^2+e^2} = \frac{v^{14}e^2}{v^{16}+v^{14}e^2+v^2e^{14}+e^{16}} \tag{2}$$

As Young (1988) had also shown, when v is close to 0.5, then the most probable candidate is the one with most votes in pair-wise comparison which in the above case is B with 66 votes rather than C with 62 votes. But the candidate with most favourable votes in pair-wise comparison is the Borda winner with Borda score being calculated as $a = 0$ and $b = 1$. However when v is close to 1, the most probable candidate is the one who wins his closest pair-wise contest with largest number of votes.

At this juncture Condorcet abandoned his statistical framework and aligned his arguments more in line with the choice theoretic framework. Condorcet argued that, as in the above example, since C defeats both A and B by a majority, it is the only reasonable choice to make. As A is defeated by both B and C, Condorcet argued that A should be treated as irrelevant and the actual comparison should be made between B and C where C should decidedly be chosen the winner. In the above argument we clearly find a tacit understanding of the principle of "Independence of Irrelevant Alternatives". He explained that the existing "First Past the Post" election system gave inaccurate results as it used too little information about the voters' preferences while Borda's method was flawed because it considered information that needed to be ignored.

2.1 *Paradox of Cyclical Majorities*

Condorcet's decision rule better known as "Condorcet criterion" landed him in the midst of a paradox namely the "Paradox of Cyclical Majorities" which emanate from the intransitivity of preference relation formed by the aggregation of individual preferences by using majority rule. A very simple example of cyclical majority is given in Table 3.

Table 3: Paradox of Cyclical Majorities

10 Voters	10 Voters	10 Voters
A	B	C
B	C	A
C	A	B

Here in the above example, there is no majority candidate according to the Condorcet criterion. It can be easily seen that A beats B, B beats C and C beats A by a margin of 20 – 10 votes. In such a situation no candidate can lay claim to be the collectively best candidate as whoever is chosen there is at least one another candidate who is preferred by a majority. Since here the groups are of equal size, the size of majority are also equal and for any candidate declared winner in such a situation the procedure would be construed as biased or discriminating. However as the groups are not always of equal size and neither are the size of the majorities, Condorcet used this feature to derive the most plausible transitive preference relation by ignoring the majority for a preference relation which is smallest in case there exists cyclicality. If there is more than one cycle, then Condorcet proposed to successively eliminate the smallest majorities. However this method is flawed as in the case of more than three candidates and in the presence of more than one cycle such a method can lead to a partial ordering rather than a full preference ordering. To understand what Condorcet might have suggested and what he might have meant let us consider the following example.

Table 4: A Pair-wise Comparison Matrix for 4 Candidates and 50 Voters.

	A	B	C	D
A	-	36	24	22
B	14	-	32	12
C	26	18	-	40
D	28	38	10	-

For elucidation we will use Condorcet's terminology, where an "opinion" is a series of pair-wise comparison of the alternatives. Each pair-wise comparison is regarded as a "proposition" and is written as A>B. An opinion is said to be "impossible" or "absurd" if its constituent propositions form a cycle such as A>B, B>C, C>A (Young 1988). Now to implement Condorcet's method we order the six propositions in decreasing size of their plurality, these are C>D, D>B, A>B, B>C, D>A, C>A. According to Condorcet's scheme the proposition C>A has to be deleted since it has the least plurality. But this does not lead us to an opinion as there is yet another cycle namely C>D, A>B, B>C, D>A, therefore we yet again delete a proposition, D>A with least majority. Likewise for the cycle C>D, D>B, B>C we eliminate B>C. Therefore we are left with propositions C>D, D>B and A>B. Here it is evident that both C and A are undominated and thus a full ordering is indeterminate.

According to the interpretation of Condorcet's scheme given in Young (1988), what he might have meant by successive elimination was that of successive reversals of the propositions with lowest majority. Accordingly, we successively reverse the propositions from C>A to A>C, D>A to A>D and B>C to C>B until there remains no other cycles and the most plausible complete ordering is given by A>C>D>B. Such a ranking obtained by using the successive reversal scheme does not however gel well with Condorcet's contention of choosing the opinion that is most probable. By most probable opinion we mean propositions which do not constitute a cycle and is supported by maximum number of pair-wise votes. There may actually be other "opinions" which have the support of higher number of pair-wise votes apart from the one arrived at through "Condorcet's successive reversal scheme".

Despite Borda's and Condorcet's seminal works on the theory of committee elections, the trail for social choice almost went cold except for some odd sparks whose contributions although no small accomplishments but failed to inspire a liveliness in the discourse. It was around the middle of the nineteenth century, owing much to the political imperatives of the time, that another lively debate arose between the majoritarian principle and the principle of proportional representation. Majoritarianism suggested that a legislator is a true representative of the voters if and only if he or she represents a majority of them while the concept proportional representation as John Adams in America and Mirabeau in

France articulated (Mclean and Urken, 1995) that the legislature should mirror exactly the constituency or district from which it is drawn and "should think, feel, reason and act like them". Of course, to produce such a similarity as a microcosm of a larger populace one gets inevitably led to the concept of multi-member constituencies. The forerunners who worked extensively on this subject were people like Thomas Hill, Thomas Hare, Carl Andrae, and H. R. Droop whose works contributed immensely towards the development of the procedure of Single Transferable vote the literature on which will be discussed in the next chapter.

3. Paradox of Arrow's Theorem

The most significant contribution in modern theory of social choice was made with the publication of Arrow's *Social Choice and Individual Values* in 1951 which led further to a flurry of activity thus firmly rooting social choice theory as a scientific sub-discipline within economics. Arrow's General Possibility Theorem (GPT, henceforth) opened up a whole new world of possibilities (or impossibilities) within welfare economics and it was quite later that political theorists actually came to terms with his results. Arrow laid down four independent conditions that were earlier implicitly assumed desirable for an aggregation rule to fulfill and had shown that there exists no such aggregation rule which can satisfy all the those four conditions together. Arrow however proved his theorem in the context of social welfare functions, so before laying bare his result we must first address some definitional issues.

Let us assume "A" to be a set of alternative social states and let there be at least three such alternatives present in A. Let "V" be a set of individuals and cardinality of V be given by $\# V = n \geq 2$. Each individual i belonging to set V has a preference ordering R_i over the set of alternatives belonging to A, the symmetric and asymmetric of part of a individual preference ordering being denoted by I_i (Indifference) and P_i (Preference) respectively. Any n-tuple of such individual preference orderings is called a preference profile ($[R_i]$). The set of all such preference profile is denoted by Π^n. A social preference ordering is denoted by R and similar as before its symmetric and asymmetric parts are denoted by I and P respectively. A "collective choice rule" (CCR) is a function which maps each preference profile $[R_i]$ to a social preference relation R. Following Sen (1970), a social welfare function (SWF) is defined as a collective choice rule *f* whose range is restricted to the set of orderings over A. Put simply a SWF is a CCR which specifies orderings for the society. The conditions laid down by Arrow can be paraphrased as below:

Condition of Unrestricted domain (U): This requires that the CCR *f* must work for every logically possible n-tuple of individual orderings over A.

Pareto Principle (P): If every individual in the society prefer x to y then the society must prefer x to y i.e., $[\forall\, i \in V: x\, P_i\, y] \rightarrow x\, P\, y$. This is also known as *Weak Pareto Principle.*

Condition of Independence of Irrelevant Alternative (I): This requires that the social choice over a set of alternatives should depend on the orderings over only those alternatives and nothing else. Let $[R_i]$ and $[R_i']$ be the two preference profiles from which two social preference relation R and R' are obtained respectively. Now, formally condition I can be defined as

$(\forall\, [R_i], [R_i'] \in \Pi^n)\, (\forall\, x, y \in S \subseteq A)\, [(\forall\, i \in N)\, [\, (x\, R_i\, y \leftrightarrow x\, R'_i\, y) \wedge (y\, R_i\, x \leftrightarrow y\, R'_i\, x)] \rightarrow [(x\, R\, y \leftrightarrow x\, R'\, y) \wedge (y\, R\, x \leftrightarrow y\, R'\, x)]]$.

Condition of Non-dictatorship (D): There should be no such individual such that her preferences will always be the preference of the society. Put in other words:

$\sim (\exists i \in N)\, (\forall [R_i] \in \Pi^n)\, (\forall\, x, y \in A)\ \ [x\, P_i\, y \rightarrow x\, P\, y]$.

Arrow's *General Possibility Theorem*: There is no SWF which satisfy conditions U, P, I and D.

Following the GPT an almost harried effort took place to find a way out of the stranglehold of the impossibility result. The crux of the impossibility result lay with Arrow's inclination for obtaining a complete social ordering, in other words the social preference relation so obtained from the individual orderings was required to be reflexive, complete and transitive. Sen (1969) proved that relaxing the requirement of transitivity to quasi-transitivity actually did away with the impossibility result for relational collective choice rules. Quasi-transitivity requires that for all x, y and z in A if x P y and y P z then x P z without requiring the transitivity of indifference[3]. In contrast to a SWF, Sen (1969) considers a family of collective choice rules which indicate clearly the "best" alternatives in every choice situation. Such a CCR termed as Social Decision Function (SDF) is defined as a rule the range of which is restricted to only those preference relations R, each of which generates a choice set C(S, R) (S being a subset of A) over the entire set of alternative A. Some results in relation to SDF and some additional properties as have been spelled out in Sen (1969) are enumerated hereafter.

Proposition 1: There is an SDF satisfying conditions U, P, I and D for any finite set A.

Two issues involving the social relation R generated by an SDF need to be noted: firstly, R is not transitive but quasi-transitive, in effect what a SDF ensures is that a choice function will always exist whatever the individual preferences are. Secondly, a SDF also satisfies a stronger version of P and D along with the other two properties.

Strong Pareto rule (Condition P*): For any x, y in A, $[\forall$ i: x R_i y and $\exists$ i: x P_i y$] \rightarrow$ x P y.

Strong Non-Dictatorship (Condition D*): There exists no individual i such that for all n-tuples of preference ordering in the domain of *f* either of the following conditions hold:

i) $\exists$ x and y $\in$ A: x P_i y $\rightarrow$ x P y;

ii) $\exists$ x and y $\in$ A: x R_i y $\rightarrow$ x R y.

Proposition 2: There is an SDF satisfying conditions U, P*, I, D* for any non-empty finite set A.

[3] $\forall$ x, y and z $\in$ A, x I y $\wedge$ y I z $\rightarrow$ x I z.

Now if the restriction of finiteness of the set A is relaxed it is found that conditions U and P are inconsistent for an SDF, however if at least one person's preference relation R_i generates a choice function then there is a SDF for the society satisfying U, P, I and D*.

Proposition 3: There is an SDF satisfying U, P, I and D*, if at least one of the individual orderings generates a choice function over the set A.

Apart from the conditions laid down by Arrow, some other conditions can also be introduced to verify if an SDF satisfies them. Important in this context is the result obtained in May (1952), which lays down conditions like Anonymity, Neutrality, Positive Responsiveness and Pair-wise Decisiveness.

Condition of Anonymity (A): A collective choice rule is anonymous iff $\forall\ i \in V$, for every permutation σ of V and for all n-tuples of individual orderings, $f(\sigma[R_i]) = f([R_i])$. In other words, anonymity requires that social preferences should be invariant with respect to permutations of individual preferences.

Condition of Neutrality (N): A choice rule is neutral if and only if for every permutation λ of A, and for every preference profile $[R_i]$ and $\lambda[R_i]$, $f(\lambda[R_i]) = \lambda f([R_i])$. Neutrality demands if two alternatives x and y respectively in A have the same relation to each other in each individuals preferences for $[R_i]$ as any z and w in A for $\lambda[R_i]$, then social preference between x and y in the first case must be the same as the social preference between z and w in the second case.

Condition of Positive Responsiveness (PR): This requires that the choice rule should respond positively to any changes in the individual preferences. This means that if the group decision is either indifference or favourable to any alternative x in A, then if all the individual preferences remain same except one individual preference which changes in a way favourable to x, the group decision should become favourable to x. Formally,

$(\forall [R_i], [R_i'] \in \Pi^n)\ (\forall\ x, y \in S \subseteq A)\ [(\forall\ i \in N)\ [(x\ P_i\ y \rightarrow x\ P_i'\ y) \wedge (x\ I_i\ y \rightarrow x\ R_i'\ y)] \rightarrow [(x\ P\ y \rightarrow x\ P'\ y) \wedge (x\ I\ y \rightarrow x\ R'\ y)]]$.

Condition of "Decisiveness": A social decision function is always decisive if and only if it specifies a unique decision (even if indifference) for every individual preference ordering.

Proposition 4: A social choice function is the method of simple majority decisions if and only if it satisfies decisiveness, anonymity (egalitarian), neutrality and positive responsiveness.

Before we state any other results regarding SDF we must define two additional rationality conditions imposed on the choice functions generated by the SDF.

Property α: It requires that if some element of subset S_1 of S_2 is the best element of S_2 then it must be best in S_1. Formally it can be written as
$x \in S_1 \subset S_2 \rightarrow [x \in C(S_2) \rightarrow x \in C(S_1)]$.

Property β: It requires that if x and y are both best in S_1, a subset of S_2, then one of them cannot be best in S_2 without the other being also best in S_2.
$[x, y \in C(S_1) \text{ and } S_1 \subset S_2] \rightarrow [x \in C(S_2) \leftrightarrow y \in C(S_2)]$.

While property α ensures that if an alternative x is chosen as best in a set of alternatives then it must remain so under contraction of the set (by dropping some alternative) on the other hand property β requires that if there are two elements chosen best in a set of alternative then one of them should not be chosen over the other in an expanded set, of which the earlier set is a subset. An example to the contrary may actually help to clarify. Consider a set of three alternatives x, y and z where x P y, y P z and x I z. In the choice over all three elements, clearly x is the best alternative since it is no worse than the other two alternatives. But considering the choice between x and z, both of them are considered best and therefore by property β should have been considered so in the choice involving all three. However this is not the case here and we have clear violation of property β.

Proposition 5: There is a SDF satisfying U, P, I and D for any finite set A, and fulfilling the requirement that each social choice function so generated by the SDF should satisfy Property α.

Proposition 6: There is no SDF satisfying U, P, I and D, fulfilling the requirement that each social choice function so generated by the SDF should satisfy property β.

Whether the consistency condition of property β is dispensable or indispensable for a social choice function is a matter of value judgement and is debatable like all other properties discussed earlier. It should be noted here that a preference relation generating a choice function that satisfies property β must be an ordering (Sen 1969, Corollary 1). Therefore a SDF that generates preference relations yielding choice functions which satisfy property β must be a SWF. Thus Arrow's impossibility result gets transformed into an impossibility theorem for SDFs if property β is imposed as a necessary condition of social choice (Proposition 6[4]).

Arrow's impossibility theorem is but just one paradox in the theory of social choice which abounds in such conflicts and dilemmas and has served the purpose of a key cornerstone in the development of axiomatic analysis of preference aggregation rules. Having elucidated on the three celebrated paradoxes in social choice we now turn to some other paradoxes which abound in the literature and queer the pitch for both theorists and practitioners alike who work with them. Some of these paradoxes may be typically inimical to certain voting procedures (e.g., additional support paradox) while some are more general feature (e.g., no-show paradox). The following section will try and explicate on some of these.

[4] See also Sen (1969), (1970) and (1977).

4. Paradoxes of Variations in Preference Profile

4.1 *No-Show Paradox*

The notion of "No-show paradox" was first articulated in Fishburn and Brams (1983) in case of a preferential voting method in connection with a wonderfully well concocted story where a certain Mr. and Mrs. Smith who despite having missed the polls was jubilant that one of their more preferred candidate had won the election, but on closer inspection found out much to their chagrin that had they reached the polling station and voted sincerely, their least preferred candidate would have won the election. Put succinctly such a paradox can occur when it is in the strategic interest of the individual voter to abstain from voting so that his more preferred candidate can win the election.

To elucidate on the paradox with the help of an example let us consider a situation where three candidates A, B and C are contesting an election in a community with 127 voters to which of course our celebrity Mr. and Mrs. Smith belong. For an election procedure the society uses the popular "Plurality with run-off" method. Like before here again Mr. and Mrs. Smith miss their election date and find out the next day that their next to best candidate B has been elected beating their least preferred candidate C in the run-off.

Table 5: No-show Paradox and Plurality with Run-off Voting

30 voters	10 voters	19 voters	22 voters	24 voters	20 voters
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

From the above table the first preference votes for each of the candidates can be tabulated as A - 40 votes, B - 41 and C – 44. Clearly none of the candidates obtain a majority of 63 votes, so candidate A having the least first preference votes gets scratched from the list and in the ensuing run-off between B and C, B beats C with a comfortable margin of 71 to 54 votes. Now if Mr. and Mrs. Smith would have made it to the polling station and had cast their vote true to their preference i.e., {A, B, C} in that order of preference, then in the first round the tally of first preference would have been A – 42, B – 41 and C – 44. Here B being the one with least number of first preference votes, gets scratched from the list of candidates and in second round run-off between A and C, C wins the election with a comfortable majority of 66 votes to A's 61 votes. Since the "Alternative vote"[5] gives the same results as does "Plurality with run-off" in case where three candidates are contesting the election, the result will hold.

The incidence of no-show paradox in a large number of voting rules undermines the very ideal of a democratic process which necessitates the act of voting by providing incentive for the voters to abstain. This necessitates for a property which a voting rule must adhere to, thereby ensuring invulnerability to the no-show paradox. Moulin (1988) defines such a property termed as the Participation property and shows that no Condorcet consistent voting function satisfies the participation property.

Participation property[6]: A voting rule is said to satisfy participation property if and only if for any given pair of situation (A, $[R_i]$) and (A, R_J) where R_J is the preference ordering of a single voter, we have then for any x, y $\in$ A,
If $f(A, [R_i]) = \{x\}$ and x is preferred to y in R_J then $f(A, [R_i] + R_J) \neq \{y\}$.
This means that if x is the winner for any preference profile $[R_i]$ and a new voter is added who prefers x to y then for the new preference profile y should not be chosen above x by the voting rule.

Proposition 7[7]: No Condorcet consistent voting rule satisfies Participation property when there are at least three candidates contesting the election.

There also exists stronger version of no-show paradox (also known as Strong No-show Paradox (SNSP)) which occurs on two instances. Under certain voting procedures it may so happen that all other things remaining unchanged a certain group of voters by

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[5] An "Alternative vote" requires individual voters to give full preference ordering over the entire set of alternatives facing them and the alternative receiving majority in first preference votes is declared elected. In case none of the alternatives obtain majority, the alternative with least number of first preference votes is eliminated and the votes are distributed to the second-in-preference alternative for these voters. This process continues until an alternative obtains majority by redistribution of votes and is thereby elected.
[6] See Perez (2001)
[7] Moulin (1988), see also Young and Levenglick (1978).

abstaining can bring about the election of their most preferred candidate. While on the other hand if the group of voters votes according to its preferences sincerely, a inferior candidate may get elected. Perez (2001) in this regard lays down two conditions namely Positive Involvement (PI) and Negative Involvement (NI), the violation of which leads to the occurrence of SNSP.

Condition Positive Involvement (PI): A voting correspondence[8] (VC) satisfies PI property if and only if for any given pair of situation (A, $[R_i]$) and (A, R_J) where R_J is the preference ordering of a single voter, we have then for any x, y $\in$ A,

If $x \in f(A, [R_i])$ and x is preferred to any y in R_J then $x \in f(A, [R_i] + R_J)$.

Condition Negative Involvement (NI): A voting correspondence (VC) satisfies PI property if and only if for any given pair of situation (A, $[R_i]$) and (A, R_J) where R_J is the preference ordering of a single voter, we have then for any x, y $\in$ A,

If $x \notin f(A, [R_i])$ and y is preferred to x in R_J then $x \notin f(A, [R_i] + R_J)$.

Before stating the two important results that Perez (2001) draws in this context, we must get over with some more definitional issue.

C1 – Domination: For a given situation (A, $[R_i]$) and two candidates x, y $\in$ A, y is said to be C1-dominated by x if and only if following two conditions hold

1) $N(x, y) > N(y, x)$[9],
2) For any $z \in A/\{x, y\}$, if $N(y, z) \geq N(z, y)$ then $N(x, z) > N(z, x)$.

C2 – Domination: For a given situation (A, $[R_i]$) and two candidates x, y $\in$ A, y is said to be C2-dominated by x if and only if following two conditions hold

1) $N(x, y) > N(y, x)$,
2) For any $z \in A/\{x, y\}$, $N(x, z) \geq N(y, z)$.

[8] A Voting Correspondence (VC) f is a function that maps a situation (A, $[R_i]$) to a non-empty subset of A, $f(A, [R_i])$. A VC which for any given situation chooses only one candidate is a Voting Function or Voting Rule.

[9] N(x, y) denotes the number of individuals who prefer x to y, similarly N(y, x) denotes the number of individuals who prefer y to x.

C2 – Quasi-domination in differences: For a given situation (A, $[R_i]$) and two candidates x, y $\in$ A, y is said to be C2-quasidominated in differences by x if and only if following three conditions hold

1) $N(x, y) > N(y, x)$,
2) $N(x, z) \geq N(y, z)$ for any $z \in A - \{x, y\}$, except for an unique element $z_1 \in A - \{x, y\}$
3) If $N(x, z_1) < N(y, z_1)$ then $N(x, y) - N(y, x) > N(y, z_1) - N(x, z_1)$.

In words this means that x outperforms y, does better than or equal to y when confronted with any other alternative except one unique z_1 in which case x's performance against y more than compensates for the difference in favour of y when both are confronted with z_1.

C2 – Domination by Pair: Given a situation (A, $[R_i]$), let us consider three alternatives x, y and z $\in$ A. Now, z is C2-dominated by pair {x, y} if and only if the following two conditions hold

1) Both x and y C2-quasidominate z in differences.
2) If $w \in \{x, y\}$ and for any $s \in A-\{x, y, z\}$, $N(w, s) < N(z, s)$ then $N(y, s) - N(z, s) \geq N(z, s) - N(x, s)$.

What this means is that apart from x and y C2-quasidominating z, the performance of any of them at the weak point of the other is enough to compensate the poor performance of the other at its own weak point.

Proposition 8:

a) No Condorcet consistent VC that weakly satisfies the *C1-Domination* or weakly satisfies *C2-Domination by a pair* satisfies PI property.
b) No Condorcet consistent VC that weakly satisfies *C1-Domination* or weakly satisfies *C2-Domination by a pair* satisfies the NI and Translation Invariance[10] properties.

[10] Translation Invariance is a necessary property of the C1-Correspondences and C2-Correspondences. It necessitates that the beating relation $N(x, y) > N(y, x)$ does not change in the translation process or in case of C2-Correspodences the difference relation $N(x, y) - N(y, x)$ does not change.

Before terminating our discussion on the no-show paradox, an example of the strong no-show paradox seems worth considering, throwing some light on how such paradoxes can pervade in real voting situation. Nurmi (2004) gives an example of SNSP in agenda voting.

Table 6: SNSP in Agenda Voting

2 voters	3 voters	2 voters	2 voters
A	B	C	C
B	C	A	B
C	A	B	A

Following the above table let the agenda of pair-wise votes be

1. A vs. B,
2. winner of 1 vs. C

If all voters vote according to their preferences then the winner is B, however if voters to the extreme right of the table who prefer C abstain from voting all other things remaining constant, C wins the election as B gets defeated in the first round and C beats A in the second round.

The example shown above demonstrating SNSP must not be confused for a similar kind paradox that is widely encountered particularly in non-monotonic voting rules and is known as "Additional Support Paradox". It is characterized by the fact that higher support for a winning candidate can make it non-winning. Like no-show paradox, additional support paradox emanates due to certain changes in preference profile, hence for some preference profiles one may not encounter these paradoxes even though the voting rule is known to be prone to such paradoxes. An illustration of the additional support paradox is given in the next section.

4.2 *Additional Support Paradox*

An inclusive democratic process seeks to involve as many people as possible so that the collective opinion so obtained (either on a certain candidate or any policy alternative) reflects the individual opinions expressed. A voting function used under such process is envisaged to aggregate the individual opinions in a fashion which ensures that an increase in the support of a candidate or a policy alternative should increase the chances of that particular candidate or alternative to get elected. Failure of a voting function to ensure this undermines the very rationale of inclusiveness.

The condition of monotonicity of voting functions in social choice theory surmises in spirit the notion of inclusiveness discussed above and can be defined as

Monotonicity Property: For a given situation (A, $[R_i]$) if an alternative $x \in A$ wins, then it must continue to do so for a preference profile $[R_i']$ where all other things remaining equal x has been placed higher in an individual preference ranking.

A voting rule failing monotonicity property is said to be vulnerable to "additional support paradox". Let us consider an example of the paradox adapted from Nurmi (1999).

Table 7: Monotonicity Paradox and Plurality with Run-off Voting

34 voters	35 voters	31 voters
A	B	C
C	C	A
B	A	B

Consider plurality with runoff system which in case of three alternatives gives the same result as the alternative vote so the results demonstrated in this example should be applicable to both the systems of voting. In the above example none of the contestants obtain a majority, so candidate C with the least first preference votes gets scratched from the list of candidates and his votes get transferred to the second-in-preference candidate, i.e. A. Therefore A defeats B in the ensuing runoff with a margin of 65 to 35 votes. However under different circumstances, assume 5 of the voters who had preference BCA now has a preference ordering of ABC, preferences of all other individuals remaining unchanged. Since support for A in the new preference profile goes up, it is expected that A should continue to be the winner. However this is not the case in our above example.

Since none of the alternatives gain majority, B with the least first preference votes (now 30 votes) gets dropped, and in the second round C beats A with a margin of 61 to 39 votes. Thus higher support for candidate A leads to its defeat in the election.

Apart from the plurality runoff procedure, multi-stage voting rules like the Coomb's rule, Nanson's method, the Single Transferable vote (STV) all are vulnerable to non-monotonicity paradoxes. However not all multi-stage voting rules are non-monotonic, a case in point is the amendment procedure used widely in legislatures of the contemporary world. In this system all the alternatives are confronted against each other in pairs and the alternative having the majority support in a given pair is matched with the next alternative until the entire set is exhausted. The alternative thus winning the last pair wise comparison, wins the vote. In this procedure however, the final outcome of the voting is dependent on the order in which the alternatives are placed in the voting agenda. If the agenda is kept fixed, then the amendment procedure is monotonic (Fishburn 1982).
Fishburn (1982) proves an important result pertaining to the general characteristics of voting rules which are vulnerable to the monotonicity paradox.

Proposition 9 (Fishburn, 1982): Assume a three alternative profile consisting of alternatives x, y and z, where x and y positionally dominate[11] z and more voters prefer x to y than y to x (x beats y in pair-wise comparison by a majority). In such a scenario, if the voting rule used is such that under this type of profile it always selects x then such a voting rule is non-monotonic.

Occurrence of non-monotonicity however, is not limited to multi–stage procedures as shown by Nurmi (2004) that single stage procedures can also be prone to non-monotonicity. One such procedure is Dodgson's method. For a given preference profile Dodgson's method elects a Condorcet winner whenever one exists, in case there exists no Condorcet winner then Dodgson's method seeks to elect the candidate who requires

[11] Positional Dominance: Let $N(x_i)$ be the number of individuals who rank x at position i (i = 1 . . . .k number of alternatives) in their preference ranking. Then x positionally dominates y (x D y) if and only if

$$\sum_{i=1}^{m} N(x_i) > \sum_{i=1}^{m} N(y_i) \text{ for all } m = 1, \ldots\ldots k-1.$$

minimum number of preference reversals to become the Condorcet winner. An example to demonstrate the non-monotonicity of Dodgson's method has been reproduced from Nurmi (1999).

Table 8: Monotonicity Paradox and Dodgson's Method

42 voters	26 voters	21 voters	11 voters
B	A	E	E
A	E	D	A
C	C	B	B
D	B	A	D
E	D	C	C

In the above example we have 100 voters and five candidates among whom there is no Condorcet winner. So by Dodgson's method we start checking for candidates who require minimum preference reversals. Here A beats all other candidates except B and hence needs 51 – 37 =14 preference reversals. Similarly B beats all other candidates except E, but since B is not adjacent to E in any of the preference ranking, the minimum number of reversals it require is 2 preference reversal of 51 – 42 = 9 voters each which means 18 reversals. Similarly for E the number of minimum reversals required is 19. C and D being beaten by every other candidate fall considerably behind. Therefore A requiring minimum of all reversals is declared the winner. Now suppose 11 voters in the right most preference ranking in Table 8 ranked A first instead of E, all other things remaining unchanged. In this case the number of reversals that A requires still remains unchanged, however, now B is adjacent to E and requires only a single reversal of 9 voters which is lesser than A's 11 preference reversals, hence B is now the winner. This shows that Dodgson's method is vulnerable to the additional support paradox.

Apart from the additional support paradox typical of voting procedures that are non-monotonic, another peculiar problem that can arise owing to the vulnerability of a voting rule to non-monotonicity is that there may be an incentive for the voters to not reveal their entire preference ordering and thereby succeed in getting elected their more preferred candidate. Such perversity originating from non-compliance of the monotonicity property is known as "Preference Truncation Paradox" which we now take up for discussion.

4.3 Preference Truncation Paradox

A voting rule that aims to use the entire preference orderings of individuals to arrive at the best candidate out of a set of candidates should be able to induce the voters to submit their entire preference orderings. Failing which a group of voters may distort the outcome of an election by indicating only its top-ranked alternative, forcing the voting rule to decide the winner without taking into account the groups' preferences in regard to other candidates. An example of preference truncation paradox in case of the alternative vote system presented in Nurmi (1999) has been adapted here, which falsifies the claim of its advocates that alternative voting system encourages the voters to provide their full preference ranking.

Table 9: Preference Truncation and Alternative Vote

33 voters	29 voters	24 voters	22 voters	17 voters
A	B	C	D	E
B	A	E	E	D
C	E	B	C	C
D	C	A	B	A
E	D	D	A	B

From the above table it is clear that none of the alternatives listed obtain majority support in terms of first preference votes. So, candidate E having the least first preference votes gets eliminated and his votes get transferred to the second preferred alternative D in the preference ranking. As there is still no majority winner candidate C gets eliminated and his votes are transferred to B, since E is already eliminated. Now A with least first preference votes (33 votes) gets eliminated. Therefore in the last round B wins defeating D with a huge margin of 86 to 39 votes.

Now consider a situation where the 17 voters who had preference ranking EDCAB indicate only their first preference E. E still being the candidate with least first preference votes gets eliminated but in this case none of his votes can be transferred to other candidates, as such in the second round D with least first preference votes gets eliminated. The 22 votes for D now are transferred to C (being the next best alternative in the preference ranking). Therefore C now has 46 votes and B with 29 votes gets eliminated transferring the votes to A, who now has 62 votes. Therefore in the last round A becomes winner beating C by a margin of 62 to 46 votes. This outcome is obviously more preferred to the earlier outcome (B) to the seventeen voters who truncated their preference.

Similarly, voting rules which are Condorcet extensions can also be prone to preference truncation paradox, an example of which Nurmi (1999) provides in the case of Copeland procedure.

Table10: Preference Truncation and Copeland's Method

1 voter	2 voters	1 voter	2 voters
A	D	C	C
B	A	B	D
C	B	A	B
D	C	D	A

In the preference profile presented above there is no Condorcet winner; hence the Copeland procedure picks the alternative which wins the maximum number of head to head comparisons with other alternatives. It can easily be found out from the preference profile that three candidates A, B and C tie with each other with each of them getting 3 votes against the other. Therefore D is declared winner with most wins in pair-wise comparison, beating both A and B and getting beaten only by C. Now suppose the voter whose preference ranking is ABCD truncates her preference and indicates only her first preference. In this case D still goes on to beat A and B, but now C also beats two candidates i.e., B and D. So both C and D tie with both having the maximum number of wins.

An important result in the context of the preference truncation paradoxes was put forward by Fishburn and Brams (1984) which claims that invulnerability to preference truncation may be inconsistent with Condorcet consistency. To provide an outline of the result, some definitional issues need to cleared as put forward by Nurmi (1999).

Definition 1: A core is the set of alternatives that are defeated by no other alternatives by a majority of votes in pair-wise comparison.
The core is therefore a generalization of the Condorcet winner. Such set may even exist for preference profiles where no alternative defeats all others. In some preference profiles the core may be empty when cyclical majorities exist.

Definition 2: A procedure satisfies strong Condorcet condition if the elected alternatives always coincide with the core whenever one exists.
This in other words means that whenever there is a non-empty core in a preference profile, all the candidates belonging to the core and only them must be elected.

Definition 3: A procedure satisfies moderate Condorcet condition if the elected candidates always form a subset of the core.
Definition 3 is a watered down version of definition 2 and requires that the elected candidates must only belong to the core. All procedures that satisfy strong Condorcet condition also satisfy the moderate condition but the converse is not true.

Proposition 10 (Fishburn and Brams, 1984): If the number of voters is at least seven, then procedures that are invulnerable to truncation paradox violate the moderate Condorcet condition and therefore also the strong Condorcet condition.

It should be noted that even when there are at least four voters, procedures immune to truncation paradox fail to satisfy strong Condorcet condition. An example in this context has been provided by Nurmi (2004) which is presented below.

Table 11: Preference Truncation and Condorcet Consistency

1 voter	1 voter	1 voter	1 voter
A	A	B	C
B	C	A	B
C	B	C	A

In the above profile both A and B remain undefeated and therefore constitute the core set. A voting procedure adhering to the strong Condorcet condition must therefore elect both A and B. Now if the left most voter in the above table truncates her preference ranking and indicates only her first preference, then A still remains undominated but B is defeated by C. Therefore a procedure satisfying strong Condorcet condition must elect only A, which is a more favourable outcome for the particular voter rather than the earlier one. This shows that procedures satisfying strong Condorcet condition are prone to preference truncation.

Paradoxes discussed in this section make clear one fact that though the monotonicity property remains one of the most obvious desirable characteristic of voting rule, many of them actually do not satisfy it. This leads to anomalous representation as well as inducement to voters for strategic manipulation of outcomes by misrepresenting their preferences. While "additional support paradox" leads to misrepresentation where higher support for a winning candidate ceteris paribus, may make it non-winning; "preference truncation paradox" induces the voters to under-report their preferences which in fact may lead to an outcome better preferred to a particular group of voters. The most dramatic of the three paradoxes discussed in this category is the "no-show paradox" which actually undermines the very rationale behind the act of voting. The "no-show paradox" induces the voters to adopt abstention as a strategy to manipulate the outcome of the election. Under stronger version of the paradox a group of voters may actually facilitate the election of their most preferred candidate without even casting the ballot.

So far in our discussion on the monotonicity paradoxes we exclusively concentrated on distortions arising out of modification or changes in preference profile. However there are some more paradoxes where counter-intuitive changes are observed owing to changes in the set of alternatives and keeping the preference profile unchanged. Such paradoxes can be clubbed as "Intra-profile paradoxes" which we discuss in next section.

5. Intra-Profile Paradoxes

5.1 *Pareto Violations*

Apart from the paradox of cyclical majorities that mostly occupies the centre stage in discussions on Condorcet social choice function, a more humble paradox but with interesting counter-intuitive implications pervade voting rules based on such binary comparison of alternatives to obtain the best candidate. Such a paradox known as "Pareto violation" occurs when the alternative elected by the head-to-head comparisons is not only the one who may be defeated by some other alternative, but may in fact be defeated with unanimity by any other alternative. The social choice axiom which is fundamental to the immunity to pareto violations is known as "Pareto criterion" and is defined as follows.

Pareto Criterion: It requires that if all voters strictly prefer x to y, then y should not be elected. (see also Condition P discussed earlier).

In case of voting rules this criterion does not say that for all voters if x is preferred to y then x should be elected, rather it says in such a case y should never be elected above x. Voting methods like amendment procedure which use Condorcet decision criterion of electing the alternative which is the majority winner in every pair-wise comparison, are more amenable to paradox of Pareto violation. An example to demonstrate the paradox is presented here.

Table 12: Pareto Violations and Amendment Procedure

Voter 6	Voter 5	Voter 4
A	B	C
B	C	D
C	D	A
D	A	B

Let us consider an amendment procedure where the agenda is set in the following order.

- B versus C,
- Winner of round 1 versus A,
- Winner of round 2 versus D.

From the preference profile in the above table, it can be seen that B defeats C in round 1, gets beaten by A in round 2 and in the final round D beats A, thus D is the winner. On a bit closer scrutiny of the profile, however reveals that C is preferred to D by all the voters. This is a clear indication of Pareto violation.

One more procedure that Nurmi (1999) enumerates in connection with Pareto violations is the successive procedure. In this procedure an individual alternative is confronted with a set of alternatives and if it obtains majority of votes it wins and no other votes are taken. If it fails to do so, it is eliminated and the alternative next in sequence is confronted with the remaining set of alternative. The procedure continues until an alternative obtains majority. In this procedure the agenda can be laid down as: a) first round: C versus {A, B, D}; b) second round: B versus {A, D}; c) third round: A versus D. Here each step in the agenda is taken provided the earlier round has not thrown up a winner. From the profile presented above it is quite clear that D wins the election in spite of being dominated by C in every voter's preference ranking in contravention to the Pareto criterion.

5.2 *Inconsistency Paradox*

The consistency property is an intra-profile condition which requires that when two disjoint set of voters, exercising their vote separately choose the same set of alternatives, then all the voters taken together should be able to choose precisely the same choice set. Let us consider that our set of voters V is divided into two disjoint sets V_1 and V_2, the preference profiles of each group of voters be denoted by $[R^1{}_i]$ and $[R^2{}_i]$ for V_1 and V_2 respectively and the preference profile for the entire set of voters V be $[R_i]$.

Consistency Property: A voting rule f is said to satisfy consistency property iff for any two situations (A, $[R^1{}_i]$) and (A, $[R^2{}_i]$)

$[f(A, [R^1{}_i]) \cap f(A, [R^2{}_i]) \neq \emptyset] \rightarrow [f(A, [R^1{}_i]) \cap f(A, [R^2{}_i]) \subseteq f(A, [R_i])]$.

This is however a weaker requirement, a stronger consistency condition can be defined as:

Strong Consistency Property: A voting rule f is said to satisfy strong consistency property iff for any two situations (A, $[R^1{}_i]$) and (A, $[R^2{}_i]$),

$[f(A, [R^1{}_i]) \cap f(A, [R^2{}_i]) \neq \emptyset] \rightarrow [f(A, [R^1{}_i]) \cap f(A, [R^2{}_i]) = f(A, [R_i])]$.

As an illustration consider a set of five alternatives {x, y, z, u, v}; the two electoral districts V_1 and V_2 with preference profiles $[R^1{}_i]$ and $[R^2{}_i]$ taking votes separately. Now if choice set for V_1 consists of {x, y, z} and the choice set for V_2 is {z, v} then strong consistency condition would require that for the entire set of voters taken together and the preference profile $[R_i]$ the choice set should be z. While weak consistency condition requires that along with z other alternatives may be included in the choice set. A violation of consistency condition then implies that for a particular voting procedure choice set for the entire population does not include alternatives belonging to the choice set of the sub-population. In single candidate elections this amounts to the fact that a candidate winning in the sub-elections in the districts may lose in election on the amalgamation of all the districts. Young (1975), proves an important result in this context which says that all social choice scoring functions are consistent. The most popular and commonly used scoring functions are the plurality rule and the Borda's method. It also puts forward an important counterintuitive result which says that all voting rules which elect a Condorcet winner whenever one exists and satisfies the anonymity and neutrality properties (i.e., non-discriminating) must be inconsistent. However in this context Young (1978) proves another theorem which shows that Kemeny's rule is the unique social choice rule which satisfies neutrality, consistency and Condorcet criterion. An example regarding violation of consistency property of a Condorcet consistent rule is presented in the following table.

Table 13: Inconsistency Paradox and Condorcet Criterion

Constituency 1		Constituency 2		
2 voters	4 voters	5 voters	5 voters	5 voters
A	C	A	B	C
C	A	C	A	B
B	B	B	C	A

In the profiles given in the above table for the two constituencies each we have a Condorcet winner in constituency 1 which is candidate C; in constituency 2 there is no Condorcet winner since there is no such candidate who has remained unbeaten by any other candidate (A beats C, C beats B and B beats A). Therefore any procedure that is Condorcet consistent and non-discriminatory should choose candidate C from constituency 1 and all the three candidates from constituency 2. So we have a non-empty intersection of the choice set of the two constituencies which means that a procedure abiding consistency property should select a candidate from the intersection set of the two choice sets when both the constituencies are combined (i.e. C). However a non-discriminatory Condorcet consistent procedure if applied in case of the combined profile should be able to choose the Condorcet winner whenever one exists, which in this case is candidate A. This shows that a social choice rule that is Condorcet consistent and non-discriminatory may not satisfy consistency property.

However voting procedures other than the Condorcet consistent ones can also be prone to inconsistency paradox, e.g. STV (Doron 1979; Fishburn and Brams, 1983), and Plurality with runoff. An example of inconsistency with respect to plurality runoff has been enumerated in Nurmi (1999) which is presented here.

Table 14: Inconsistency Paradox and Plurality with Run-off Voting

East			West		
35 voters	40 voters	25 voters	40 voters	55 voters	5 voters
A	B	C	C	B	A
C	C	B	B	C	C
B	A	A	A	A	B

Let us consider that an election constituency is subdivided into two sub-constituencies (namely east and west) of hundred voters each, whose preference profile is given in the above table. In the eastern constituency candidate C with the least first preference votes gets eliminated in the first round, while B goes on to defeat A in the second round thereby winning the election. In the western constituency A with minimum first preference votes gets eliminated with B winning the election beating C. Now if we amalgamate the two constituencies it would be expected that B should go on to be the winner, however it is the candidate C who goes on to win the election. Clearly this is a case of violation of the consistency property. As have been stated before, plurality runoff being similar to STV in case of three candidates, the result holds also in case of STV.

5.3 *Choice Set Variance Paradoxes*

Paradox of this kind is encountered when the preference profile is kept constant but various subsets of the set of alternatives are considered. The notion of this paradox arises from the failure of some choice rules to satisfy the requirement that an alternative considered in a set should continue to be considered so in all its proper subset. This requirement is known as the α condition, Chernoff property or contraction consistency (Sen, 1970; Sen, 1977). Satisfying the α condition is very uncommon among voting procedures and an example demonstrating how the paradox works is illustrated in the following table adapted from Nurmi (1999).

Table 15: Choice Set Variances and Borda's Rule

3 voters	3 voters	3 voters	2 voters	2 voters	1 voter	1 voter
A	B	C	C	D	C	C
B	A	D	B	A	D	B
C	C	A	D	B	B	A
D	D	B	A	C	A	D

The Borda scores calculated from above preference profile is given as A - 23, B - 24, C – 27, D – 16. Therefore the collective ranking is CBAD. Now suppose D is removed from the preference profile and scores are computed as A – 16, B – 15 and C – 14. This means that as we consider the subset the collective ranking gets reversed from CBA to ABC. This means Borda's method fails to satisfy the α condition. Similar results from the above profile can also be obtained by using the plurality rule. An interesting result in this respect is brought out by Saari (1989) which says that using positional voting procedures on a set of alternatives, the collective preference ranking so obtained cannot be consistent with the ranking resulting from application of such a procedure in any subset of the superset. This effectively means that no positional procedure satisfies the α condition of consistency.

6. Conclusion

The discussion on the paradoxes outlined above revolves around a notion of fairness of a democratic process which requires an aggregation procedure to abide by certain norms whose purpose is to ensure that the collective choice arrived at mirrors the opinion of the society. From this notion itself, germinates the requirements of non-dictatorship, the Pareto criterion, the principle of independence of irrelevant alternatives put forward in Arrow's *general possibility theorem.* While May (1952), characterizing the method of majority decisions, put forward concepts of anonymity, neutrality, positive responsiveness (monotonicity). A host of such properties have been stipulated in social choice theory which intends to provide a bulwark against the inconsistencies that have been found to arise with the changes in preference profiles or with variations in the choice set. Unfortunately a handful of these seemingly innocuous properties are mutually consistent thereby leading to the fact that there are only few aggregation procedures which satisfy some of these properties taken together. A dramatic example of this is the Condorcet criterion which requires that a voting rule should be able to choose the majority winner whenever there exists one but it has also been seen that Condorcet consistent voting rules are incompatible with the monotonicity property, the participation property and also the consistency property, all of which individually are desirable. While all scoring functions satisfy the condition of consistency, but may fail to satisfy the α condition of consistency which follows from the impossibility result of Saari (1989), that no positional voting procedure is α consistent. Resolution of such incompatibilities therefore still rests on the value judgement of what is more desirable and what is less.

III. Majoritarian Principle and Proportional Representation – An Issue of Representativeness

1. Introduction

In classical theory of social choice initiated by Arrow (1951) the approach towards evaluating alternative voting schemes is by laying down a set of axioms considered desirable for a voting procedure to fulfill, then evaluate each of the schemes in terms of each of the desiderata they satisfy or fail to do so. But an alternate genre of analyzing voting schemes puts more emphasis on the actual operation of the procedure and properties like representativeness, whether a voting rule encourages participation of the electorate, discourages fragmentation of political parties or the kind of government they foster. As Sen (1995) puts it, the rationale behind the use of axiomatic analysis to assess voting procedures rests primarily on the belief that merely taking a synthetic view of a procedure may not give a complete understanding, and this could be complemented by the use of pre-specified requirements of good performance for evaluation of a voting procedure.

Levin and Nalebuff (1995) in the context of choosing a representative electoral procedure raise a host of questions like "what is meant by representativeness?" or "how does one go about choosing a representative outcome?" Put simply the contention about the choice of an electoral procedure revolves around the issue of choosing a single winner or multiple winners or just provide a ranking of the contesting candidates or if the procedure favours candidates with strong support from an organized group of minorities or candidates who have a general acceptability but are favourite of few or none. To have plausible answers to the above issues one must delve into the age old debate of the majoritarian principle (MP) and the principle of proportional representation (PR). The majoritarian principle advocates that the candidate with support of the majority (whether simple or absolute) of the electorate is the best candidate to represent the constituency. The idea of proportional representation puts forward that the legislature should as far as possible secure the different shades of opinions within the society, i.e. it should be a true reflection of the society and its various concerns. Both the notions of representativeness are mutually inconsistent. While the first one inevitably underscores the utility of single-member constituencies, the latter invariably leads us to multi-member constituency. Majoritarian system emphasizes effective governance by enabling a decisive majority in the legislature and preventing fragmentation of political parties. The single vote method commonly used in majoritarian systems induces the voter to cast his vote in favour of either one of the two parties or candidates whom she thinks can come closest to win the election. Therefore vote for other parties on the fringe are considered wasted, severely restricting competition by undermining the representation from minority society. In a highly fragmented society a situation may arise that majority of seats in the parliament may be held by a party with only minority support. On the other hand the system of proportional representation assures representation to all shades of opinions but in sharp contrast to the majoritarian system leads to fragmentation or multiplication of political parties in the legislature thus undermining effective and stable government by fostering coalition building. Black (1958) provides a possible justification of proportional representation which is in his own words, "what is wanted in the modern world is not 'strong' government but rather 'weak' government and a respite from the present turmoil of government activity."

In our subsequent sections we will take up some popular or widely used voting procedures from both the categories of majoritarianism and the PR system to explicate on the issues outlined above. Moreover as a follow up to the comparative analysis of the two systems of voting we discuss some pertinent issues pertaining to the definition of representativeness.

2. The Majoritarian Principle

According to Norris (1997) a majority of the democracies in the world use majoritarian voting systems and the most popular scheme used is the plurality rule or "first past the post election". Some of the leading democracies like U.S.A., U.K., Canada, India and many other Commonwealth states use this system of election due to its implementability in large electorates and simplicity in tabulation as it requires the candidate to achieve only a plurality above others to be elected. There are also other majoritarian voting rules which use absolute majority as their decision criterion like the "Plurality run-off" used in French presidential election or the "Alternative vote" which is used in elections to Australian House of Representatives and also the Irish presidential elections.

2.1 *The Plurality System*

Under this system of voting candidates are not required to overcome any minimum threshold of support neither are they required to gain an absolute majority, it is only sufficient for a candidate to get a single vote more than her closest rival to get elected. Therefore use of such a system in a country like India, where for election to the lower house of the parliament the country is divided into 545 constituencies, a party may actually capture all the seats in the parliament given it has just 545 supporters more than its closest rival spatially distributed across all the constituencies. The spatial distribution of voters in this system of voting is critical to the outcome. For example, if the critical 545 supporters of the above party are concentrated in a single constituency the concerned party actually wins only this constituency and ties in all others with its closest rival despite having polled the highest number of favourable votes. On the other hand for minority groups the reality seems to be different, as for these groups who cannot bank on a national support it pays off to be spatially concentrated in only a few constituencies to have representation in the parliament. An interesting example in this context is provided in Norris (1997) where he recounts the experience of 1993 Canadian election where the Progressive Conservative party secured 16.1 percent of the total votes polled but managed to elect only two MPs while the Bloc Quebecois managed to get 18.1 percent of votes polled but got elected 54 MPs. A more dramatic twist in this example is the New Democratic Party won fewer votes (6.6 percent) than the conservatives but managed to bag nine seats in the parliament. The primary reason for the above anomalies originates from the inclination of the plurality system to exaggerate the share of seats for the leading party to produce a strong majority in the parliament while undermining the minority parties whose support base is spatially dispersed, the underlying principle being promoting effective governance rather than representing minority views.

2.2 *Plurality Run-off Voting*

In this system of voting it is required that candidates should receive an absolute majority to get elected. If in the first round none of the candidates achieve absolute majority, a run-off is held between the two candidates who received the highest percentage of votes in the first round. In plurality run-off it may so happen that in case of a run-off taking place the candidate garnering maximum support in the first round may actually be defeated in the run-off as the candidates who have been eliminated in the first round may actually rally their support behind another candidate in the second round. Plurality run-off fosters pre-election coalition building and aims to consolidate support behind the victor. An example of the Russian presidential election may serve to highlight the working of a run-off voting (Norris, 1997). In 1996 Russian presidential election the top contenders in the first round of voting were Boris Yeltsin with 35.3 percent of votes, with Gennadi Zyuganov close behind at 32 percent and Alexander Lebed with 14.5 percent. In the run-off between the top two contenders Yeltsin and Zyuganov, Lebed rallied his supporters behind Yeltsin giving him a comfortable victory margin of 53.8 percent votes against Zyuganov's 40.3 percent.

2.3 *Approval Voting*

Approval voting does not require voters to rank the candidates in order of their preference. Here voters simply indicate the candidates they approve or disapprove, i.e. they select a subset from the set of candidates. The candidates are ranked on the basis of the number of voters who approve them. Approving all the candidates on the ballot is equivalent to not voting, as there is no differential impact on any of the candidates. Approval voting is almost equivalent to the plurality voting method where voters indicate just their most preferred alternative. In case of approval voting voters can vote for more than one candidate, the number of votes assigned to each voter being equal to the number of candidates on the ballot or in some cases it may be restricted to a number less than the number of candidates.

2.4 *Alternative Vote*

The "Alternative vote" is another majoritarian voting system which is used in elections to both the Australian House of Representatives and Irish presidential election. Here voters are required to rank the candidates in order of their preference and the candidate receiving absolute majority, either in the first round or in subsequent rounds by way of vote transfers, is declared elected. In the first round of vote counting if a candidate receives absolute majority in first preference votes he is declared elected and the process stops. However if no candidate gets majority first preference votes then the candidate with the least first preference vote is eliminated and his votes are transferred to the second preferred candidates. In the second round again the votes are tabulated and if a majority winner emerges he is declared elected otherwise the same process is repeated until there emerges a majority winner. The alternative vote is the single winner variant of single transferable vote (STV) which is used in multi-member constituencies. This voting method like all other majoritarian system translates a narrow lead into a decisive majority in the legislature and systematically discriminates against the minority social groups who are at the bottom of the poll to promote effective governance.

The voting systems based on majoritarian principle are found to be severely deficient in regard to ensuring minority representation; however what they promote is transparency and accountability by establishing a direct relationship between the candidates selected and their constituency. Proponents of majoritarian principle argue that it is this accountability that enables the constituency to exercise control over the fate of deliberations and decisions in a democracy.

3. Proportional Representation Systems

The PR system although not as popular as majoritarian voting procedures owing to its complexity, is still used in quite a large number of countries practicing parliamentary democracy. Worldwide around 57 out 150 countries use the party list system in multi-member constituencies to elect their representatives to the parliament. A PR system works on the philosophy of providing a representative legislative body which will be a microcosm of the entire population. In other words a proportional representation seeks to ensure for each individual in the constituency a representative who would as closely possible approximate the ideals and aspirations of the individual. An immediate fallout of achieving proportional representation through multi-member districts is that it clutters the visible link between a representative and his constituency. In the following sub-section we examine two voting schemes widely used to ensure proportional representation.

3.1 *Party Lists System*

Party lists system works on the principle of apportioning seats in the legislature on the basis of the proportion of votes obtained by the competing parties out of the total votes cast. A simple version of how the system works can be captured through the following example. Let A, B and C be three parties competing in a four member constituency. Let their respective vote share be 60 percent for A, B getting 22 percent and C with 18 percent. Now if the four seats are to be divided in proportion of the votes received by each party then the share of each of A, B and C stands at 2.4, 0.88 and 0.72 respectively. Therefore it seems reasonable to apportion 2 seats to party A being the closest integer to the proportion obtained, 1 seat each to both B and C along the same logic. Another way of allocating the seats is by finding out the relative strength of each party. Relative strength of the votes can be expressed as A: B: C = 3.3: 1.2: 1. Following the relative strength approach party A is allocated 3 seats and party B has 1 seat while party C gets nothing. Black (1949) claims that the relative strength method of arriving at proportion of seats may be better than the method of dividing the seats strictly in terms of the percentage of vote. This claim may be erroneous. Considering the above example it becomes apparent that in case of determining relative strength of votes, if the distance between largest party and the smallest party is big enough then the smallest party may be deprived of any seat at all. In the above example for calculating relative strength of party A with 60 percent votes garner 3 seats while party C inspite of being close enough to the second largest party does not get any seat at all, only remaining seat being awarded to party B. In some cases the method of relative strength may actually approximate to "winner take all" principle of the majoritarian system of voting, thus penalizing heavily the minority parties. Consider the election in a five member constituency where five parties A, B, C, D, and E contest the election. Let the percentage of votes obtained by each party be given as A 44 percent, B 16 percent, C 15 percent, D 14 percent and E 11 percent. The relative strengths of vote for each of the parties are represented as A: B: C: D: E: = 4.4: 1.5: 1.4: 1.3: 1. Apportioning of seats by relative strength means that party A here captures four seats out of five and B gets one seat while the remaining three groups goes without any representation in the constituency.

Party lists in the party list system of voting may be *open* as in countries like Norway, Finland, The Netherlands and Italy where voters can express their preferences for particular candidates in accordance to the party they vote for. Party lists may also be *closed* as prevalent in Israel, Portugal, Spain and Germany, where voters' mandate is restricted to the party and the ranking of the candidates is decided by the parties. A closed party list system however puts the discretion in the hands of the respective parties to nominate the candidates to their allocated seats, while the open list system puts onus of selecting candidates directly in the hands of the people and so may be inferred as more representative.

The party list system may vary, with different formulae for apportionment of seats being used in different countries. There are two methods of allocating votes to seats, one is the *highest averages* method and the other is the *largest remainder* method. The *highest averages* method requires the number of votes for each party to be divided successively by series of divisors and the seats are allotted to parties that secure the highest resulting quotient, up to the total number of seats available. A widely used method of dividing the votes is the d'Hondt formula which is given as

$$H = \frac{V}{S+1}$$

where V = number of votes; S = number of seats the party has been allocated so far (initial value being zero for all parties).

Another method that is widely used is the Sainte-Lague method which divides the votes with odd numbers, the formula being given as

$$SL = \frac{V}{2S+1}$$

where V = number of votes; S = number of seats the party has been allocated so far (initial value being zero for all parties).

A modified Sainte-Lague method uses 1.4 as the first divisor but is similar to the pure Sainte-Lague method.

The *largest remainder* method uses a minimum quota to divide up the number votes in order to arrive at the number of seats to be allocated to each party. The quota signifies the minimum number of votes required by each party to hold a seat. There are different ways to calculate a quota, the simplest way being Hare quota which is calculated as

$$q_H = \frac{V}{S}$$

where V = total number of votes polled; S = number of seats to be allocated.

Another popular method of calculating the quota is the Droop quota which is given by

$$q_D = \frac{V}{S+1} + 1$$

where V = total number of votes polled; S = number of seats to be allocated.

In the case of *largest remainder* method the seats are allocated to each party in accordance with the quotient obtained by dividing the number votes secured by the party by the quota obtained through any of the above formulae. Using the quotient to allocate the seats usually leaves some seats unallocated. The parties are then ranked on the basis of their remainders. An additional seat is allocated to each of the parties in the order of their ranking on the basis of remainder, with the party having largest remainder getting the first additional seat and so on, until all the seats are exhausted.

3.2 *The Single Transferable Vote*

The single transferable vote (STV) is another much vaunted proportional representation system which is used mostly in countries with strong English influence like in Ireland, Australia and some other commonwealth countries. STV requires the voters to provide a complete preference ranking on the list of candidates. A quota is calculated based on the total number of votes polled and the number of candidates to be elected. The quota is designed such that each candidate elected should hold a share of votes which would preclude any other candidate from obtaining the quota. Accordingly the quota is calculated using the popular Droop's formula, which means if s is the number of candidates to be elected then no more than the desired number of winners (s) can achieve the desired quota (Levin and Nalebuff, 1995). There is however one objection to dividing the total number of votes by (s + 1) as this leads to creation of (s + 1) equal sized groups where only s groups gain representation and one group of the same size goes without it (Tideman, 1995). With the quota decided the vote counting process looks for candidates whose first preference votes exceed the quota. If there exists such a candidate then he is deemed to be elected and his surplus votes are transferred to the second preference of the voters. The surplus (r) is calculated as the difference between number of votes received by the candidate (v) and the quota. However, the surplus is not transferred at its full value and is reduced to a fraction[12]. This weighting ensures that the sum of all candidate's votes remain equal to the number of votes originally cast, this in a way guarantees that there are exactly s winners in the election (Levin and Nalebuff, 1995). The name of the candidate so selected is scratched from the list of preference rankings, the quota is being re-calculated and after the votes have been transferred we again look for candidate who overcomes the quota. If there is such a candidate then the process outlined above is repeated until all the seats have been filled. However in any round if none of the candidates can achieve quota then the candidate with least first preference votes is eliminated and the votes of the eliminated candidate is transferred at full value to the candidates second in preference, followed by the above process again being repeated until all the seats have been filled. An important feature of STV is that a united minority can elect candidates in proportion to the size of the minority thereby ensuring proportional representation. Dummett (1984) has shown that STV satisfies the condition of "proportionality for solid coalitions". A solid coalition for a set of candidates C is the set of voters who rank all candidates in C above all other candidates. Therefore any such solid coalition attaining at least 1/(s+1) of the electorate can be sure of electing at least

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[12] The weighted surplus is calculated as $w = \frac{r}{r+q}$; where r is the absolute value of surplus to be transferred and q is the quota.

one representative from its preferred set of alternatives. Moreover STV does not place any premium on the organizing capability of any minority group to rally behind a single candidate as every individual belonging to that group may rank the candidates from its preferred set of alternative in any order but above other candidates thereby ensuring that at least one of them is elected. Thus STV also avoids gerrymandering with electoral districts to ensure minority representation.

The STV has been criticized on many counts; one of the basic criticisms of STV coming from Black (1958), where he lays two conditions for any PR scheme to satisfy. These are a) it must be a mathematical scheme stating a unitary principle and not merely an arithmetical rule of thumb; b) it must take into account the entire preference schedule of each voter. Neither of these conditions the STV satisfy, as it does not follow any unitary mathematical principle but rather is a set of practical rules of vote counting, moreover when considering second or lower preferences it does not follow any set principle but does so in an erratic manner. The criticism can be summarized in Black's own words as "the Droop quota is the strong feature of the single transferable vote, but apart from these there is a great deal in this method of election that are without reason and merely wooden" (Black, 1949). An important drawback of STV is that it is non-monotonic which means that under this scheme of voting a winning candidate can lose an election as a consequence of increased support (Doron and Kronick, 1977). Apart from non-monotonicity, other drawbacks to which STV is prone to are a) it is Condorcet inconsistent, i.e. a candidate who can defeat every other candidate in pair-wise comparison may not get elected; b) fails to satisfy consistency as a candidate winning in each sub-constituency separately may lose the election when all the constituencies are taken together (Fishburn and Brams, 1983).

Proponents of STV extol its virtues in dealing with wasted votes by its vote transferal mechanism. The transferal mechanism ensures that the vote of an individual never goes wasted; that is, it always brings about the election of one candidate or other. Dummett (1997) has been particularly scathing in his criticism of the STV in this regard. Dummett argues that a voter whose most preferred candidate is eliminated and his vote contributes to the election of a candidate whose ranking is much lower in the voter's preferences will have little consolation in finding out that his vote has not been wasted. A problem with the process of assessment in STV is that it treats the second, third or lower preferences of some votes at par with first preference votes of some other. It takes account of only the first choices of those voters who remain in contention till the final stage, disregarding the strong support for a candidate in terms of second preference votes while treating second preference votes at par with first preference votes for voters whose first preferred candidate has already exited (either by attaining the quota or getting eliminated). Thus while first preference votes in small numbers are disregarded quickly, candidates who just marginally attain the quota may pick up large number of votes in subsequent rounds by way of redistribution and go on to win the election. This erratic nature of assessing votes lends STV a particular character which Dummett (1997) calls *quasi-chaotic*. This characteristic manifests itself in a way that small changes at the beginning of an assessment process may get magnified into huge changes at a later stage. A small change in the beginning may cause different candidates to be eliminated or selected, thereby resulting in a big variation in redistribution of votes in subsequent stages leading to unwarranted changes in the electoral outcome. This unpredictability in the outcome leaves the voters in the dark about the possible fallout of his voting in one way or other. Dummett (1997) provides an example to demonstrate the quasi-chaotic nature of STV. Consider a four member constituency consisting of 99,995 voters and eight candidates (A to H) who contest the election. The quota using Droop's method is calculated as 20,000. Dummett assumes that there is a group of 100 voters whose ranking of the candidates is given as A D B C G F E H. Now if this group of 100 voters changed their mind and reverse the rankings of A and D, i.e. gives D first preference and A second then intuitively four cases may arise a) if A was the winner initially then he loses and D becomes the winner; b) both A and D loses the election; c) A continues to win the election and D maintains status quo; d) both A and D win the election all these happening without affecting the other candidates. However in case of STV this may not be the case always. Consider the following vote tabulation table using STV for the above mentioned constituency.

Table 1: Vote Tabulation in STV

A	B	C	D	E	F	G	H
20,004	19,000	12,046	9,595	9,850	10,050	9,670	9,780
-	19,000	12,046	(9,599)	9,850	10,050	9,670	9,780
-	*23,700*	16,945	-	9,850	10,050	9,670	9,780
-	-	*20,645*	-	9,850	10,050	9,670	9,780
-	-	-	-	9,850	10,050	10,315	(9,780)
-	-	-	-	(10,050)	14,850	15,095	-
-	-	-	-	-	19,300	*20,695*	-

In the above table figures in *italics* indicate candidate having attained quota, while figures in parentheses indicate candidate being eliminated. From the above tabulation process it is clear that candidates A, B, C and G are elected.

Now if 100 voters alter their preferences between A and D the vote counting process changes as shown in Table 2.

Table 2: Change in Vote Tabulation in STV with Slight Change in Preferences

A	B	C	D	E	F	G	H
19,904	19,000	12,046	9,695	9,850	10,050	(9,670)	9,780
19,904	19,000	(12,046)	12,095	12,250	12,349	-	12,351
19,904	19,000	-	*20,095*	15,250	12,349	-	13,397
19,934	19,020	-	-	15,275	(12,359)	-	13,407
19,993	19,320	-	-	*22,275*	-	-	18,407
19,993	19,520	-	-	-	-	-	*20,482*
20,393	19,602	-	-	-	-	-	-

Now the successful candidates are A, D, E and H. Comparing the tabulation process and the outcomes from Table 1 and 2 we can note that elevation of D in the preferences of 100 voters has enabled him to win the election while maintaining the status quo for A. What further should be noted is that such a minor change in preferences between A and D for just a small group of voters has made three candidates B, C and G lose their seats, none of whom were involved in the change in preferences. Such waywardness in the outcome emanates from the fact that both the tabulation process has followed different path of redistributing the votes leading to very different outcomes. In Table 1 selection of candidate A in the first round and subsequent elimination of D leads to reallocation of votes from D to B and C. While in Table 2 postponement of election of A in the first round and consequent elimination of G, leads the tabulation through a different process of redistribution thereby arriving at a very different outcome than what was envisaged.

The discussion of the voting procedures in the above sections underlines the differences in the theoretical underpinnings of the different procedures and the way they address the issue of representation. However a formal articulation of the issue of representativeness of a voting procedure remains desired. Chamberlin and Courant (1983) make a headway in this direction by bringing in the issue of representation into the realm of traditional social choice theory in attempting an axiomatisation of representativeness. In delving into the issue of representativeness they distinguish between two distinct points of view of a representative democracy. While one point of view looks at representative democracy as a working model of direct democracy and is most successful when it generates deliberations and decisions which are close to the ones generated in a direct democracy, the other view asserts that a representative democracy is better than direct democracy as it enables more informed deliberations and decisions. Both the views seek to evaluate representative democracy using the concept of direct democracy as a benchmark. The characteristic of a direct democracy is that it seeks full participation of all citizens in all the political processes and it is this virtue that proportional representation systems seek to uphold.

4. Defining Representativeness

Chamberlin and Courant (1983) while bringing forth their interpretation of the "representativeness" of a committee or legislature[13] raise two questions each of which gives rise to two very different interpretations of representativeness. The first question put forward is "How well does a committee represent you?" This approach leads to a preference ordering over the committees starting from committees having large number of individuals' most preferred candidates to committees with members who are less preferred. This preferences over committee approach focuses on the outcomes as well as the nature of committee selected and a committee selection procedure that takes

[13] Chamberlin and Courant (1983) focus their discussion on small committee elections, but we will use committee and legislature interchangeably as our primary focus is elections with large electorates.

cognizance of such preferences usually give more importance to the majority's right to prevail in decision making. In this approach minority's right to representation is subverted by the importance given to majority's preferences. This approach leads us to a concept of representativeness which is regarded as representativeness in decision making. The second approach poses the question "How well does your representative in the committee represent you?" The approach focuses on, not the preferences over committees but preferences over the candidates, primarily emphasizing on the representation of the values and ethos of the individual in the deliberations of the committee rather than emphasizing on the importance of it influencing the decisions of the committee. This in other words can be termed as representativeness in deliberations.

Chamberlin and Courant (1983) for formalization of "representativeness" adopt the second approach as the normative basis for a representative democracy which embodies direct democracy. The adoption of second approach towards representativeness allows an individual a right to a representative and not to a representative committee, the essence of which becomes the key cornerstone for the development of the *axiom of representativeness*. The measure of representativeness for an individual is based on the following two assumptions: a) the higher the rank of the representative in the individual's preference ordering the more representative he is. In other words committee that contains the first preference of an individual is considered to be more representative than the committee that contains his second or third choice. Similarly, the committee that contains the second preference of an individual is considered more representative than the committee that contains none of his first or second preferred candidate; b) the preferences of the individual regarding other members of the committee does not affect the measure of representativeness of a committee as a deliberative body.

4.1 ***Axiom of Representation*** (Chamberlin and Courant, 1983)

To present the axiom of representation we must first lay down some definitions and notations. Let V be a set of individual voters indexed by $i \in \{1, 2, \ldots\ldots, v\}$ and let S be a set of candidates, indexed by $j \in \{1, 2, \ldots\ldots, m\}$. A committee of size k ($1 \leq k \leq m$) is to be chosen from the set of candidates S. Let C be the set of all possible committees of size k and C' be the set of non-empty subsets of C. Individuals have linear preference orderings over the elements of set S. Preference profile for individual preferences is given by $P = (P_1, P_2, \ldots\ldots P_n)$, where P_i is the preference order of individual i over the elements of set S. Let Π be the set of all P. The individual preferences over the elements of set C are denoted by $P' = (P_1', P_2' \ldots\ldots P_n')$. P' is assessed directly by the individual or is derived from P indirectly[14]. Let Π' be the set of all P'.

Representation Axiom: For all $c, c' \in C'$ and for any $P' \in \Pi'$, $c \; P_i' \; c'$ if and only if there exists a $j \in c$ such that $j \; P_i \; k$, for all $k \in c'$.

According to the axiom a committee c is more representative for individual i than committee c' if there exists a candidate j in c such that j is preferred to all the candidates in c'.

The axiom as articulated lays down a normative benchmark for an individual to judge the most representative committee from a set of all possible committees and takes the analysis further by putting forward a committee selection function based on a modification of the Borda's rule that chooses a maximally representative committee for each individual in the society. This marks an important step in establishing a point of reference for evaluation of how representative the committee is from the society's point of view or how closely a proportional representation system approximates direct democracy.

Taking cue from the *axiom of representation* it can be argued that, from the society's point of view an ideal point in the selection of a committee is when every individual in the society is represented in the committee by his most preferred candidate. This is more akin to the notion of direct democracy discussed above, where every individual represents himself in the deliberations and decision making of the society. However apart from the question on practicality of implementing this in large electorates, it may also be the case that representative democracy may in some cases be preferred to direct democracy, especially when individuals may find someone amongst themselves more

[14] P' is induced from P by the two assumptions guiding the measure of representativeness of the committees for an individual.

capable of articulating their ideals and values in the process of deliberations. Therefore in a representative democracy, in accordance with the ideal scenario discussed above, an ideally representative committee is one where every individual in the society is represented by his most preferred alternative. Such a committee would consist of members whose total tally of first preference votes exhaust the total number of votes polled in the election. This means that the members of the committee have been selected exclusively on the basis of the strength of the first preference votes they have obtained and there is no candidate outside the committee who has been given first preference by any group of voters. A measure of representativeness of a committee therefore can be interpreted as the distance of a committee selected by any committee selection function from the ideally representative committee. Monroe (1995) uses a measure of misrepresentation to arrive at a proportional representation system which he calls "Fully Proportional Representation" (FPR) in order to select the most representative committee. To measure the misrepresentativeness, the distance of a candidate from the most preferred candidate in a voter's preference ordering is calculated. This is very similar to the inverse Borda rule, where the most preferred candidate has a score of misrepresentativeness as zero and the score is higher as the candidate moves down the voter's preference ordering. Now using such a score the FPR divides the electorate into as many equal sized groups as the number of candidates to be elected. Now to attain equal sized groups votes from larger groups are transferred to smaller ones in such way that voters whose votes are transferred suffer minimum misrepresentation. Both Chamberlin and Courant (1983) and Monroe (1995) use different modifications of Borda rule as a measure of representativeness to select a representative committee; however both the approaches suffer from weaknesses similar to that of Borda rule in terms of manipulability by voters.

5. Conclusion

The debate on the issue of representativeness in the context of majoritarian principle and the PR system highlights the differences in its interpretation. While the former underlines the necessity of representation in decision-making, the later emphasizes representation in deliberations within the society. Various criticisms have been labeled against each of the interpretations by the other camp and while the situation looks irreconcilable, the paradox here lies in the fact that an ideal democracy imbues both the interpretations of representation. A significant step towards formalization of the definition of representativeness as enumerated above has been undertaken by Chamberlin and Courant (1983) and has been extended further in formulating a measure for representativeness to select the most representative committee. However as pointed out such measures are highly manipulable, much remains desired in the form of taking up rigorous analysis in defining an objective measure of representativeness and examining it under the light of existing standard axioms of social choice.

IV. Conclusion

The questions that have lurked beneath the entire preceding analysis and are yet to be overtly stated and addressed are "why is a voting system so necessary for a functioning democracy?" or "why can't there be a single individual or a group of individuals who having the information on preferences of the society go on to decide on its fate?" Given the revolutionary strides in information technology, gathering such information may be more cost effective rather than the society deciding for themselves through ballots. Even more paradoxical at the micro level is the individuals' decision to vote. While casting his vote, an anonymous voter cannot reckon if the benefit accruing from the outcome of the election is at least going to offset his cost emanating from his decision to vote, on the other hand if the outcome of the election is a foregone conclusion for him then his decision to vote or not to vote becomes meaningless.

Nevertheless an election seems to be a necessary exercise for a democratic society. The underlying purpose of voting is that it allows an individual to reveal his/her preferences over a set of alternatives and the preferences of all the individuals when aggregated through a preference aggregation rule, one arrives at the preference of the society as a whole. It is from this point onwards that social choice theory takes off with Condorcet's putting forward of the majority criterion as the best way to approximate social preference, to Arrow's laying down of restrictions (reflexive, connected and transitive) on individual as well as social preference orderings and thereby showing that with these restrictions there exists no preference aggregation rule which satisfies properties like unrestricted domain (U), Pareto criterion (P), independence of irrelevant alternatives (I) and non-dictatorship (D) taken together. The desirability of these properties emanates from a perception of fairness that a preference aggregation rule is required to exemplify. These properties require a voting rule to behave in a manner that would lead to an outcome that would be most representative of the preferences of the society. For example, the *Pareto criterion* (P) requires that if all individuals in the society prefer one alternative (say x) over another (say y), then a voting rule should not choose y over x. Similarly, the principle of *independence of irrelevant alternatives* (I) says that social preference orderings over a set of alternatives should depend on the individual preferences orderings over that set of alternatives only. The property of *non-dictatorship* (D) requires that there should not exist an individual whose preferences should dictate the preferences of the society. Another important property that should find mention in this context is the *monotonicity property* which envisages that for a voting rule an increased support for a winning alternative *ceteris paribus,* should not make it non-winning. Each of these properties exemplify a notion of fairness in the outcome of a preference aggregation process and their violations lead to counter-intuitive results, some of which have already been demonstrated.

However fairness of an outcome and its representativeness are two very distinct concepts which can overlap only when individuals in a society are homogenous in terms of their predilections over a set of alternatives. Individuals in a society are in contrast heterogeneous in terms of their preferences and more so in a society fragmented ethnically or ideologically, therefore individual preferences over a set of alternatives may be guided by allegiance to variety of identities to which an individual adheres. While for some issuing food coupons by the government may be very close to heart, for others increasing emission of green house gases may be a cause of concern. Therefore it seems only logical that apart from ensuring fairness, a preference aggregation rule should also take care of representativeness of an outcome.

It is in the context of representativeness of an outcome that we enter the debate between the majoritarian principle and the principle of proportional representation. The majoritarian principle as the name suggests gives precedence to choice of the majority group in a society marginalizing the issue of minorities. The argument for majoritarian voting almost toes the line of Condorcet's argument that majority at majority of times chooses right. The majoritarian principle has been aptly criticized in the words of Sterne (1871)[15] which are as quoted – "This scheme proposes that after the majority have elected their representatives, a majority of these representatives shall make the laws; now add to the minority excluded from all representation who may form almost one-half of the

[15] Cited in Chamberlin and Courant (1983).

voters, that number of the majority who are represented by the dissenting members of the legislative body and you place the law-making power into the hands of the representatives of the minority of the people." Proportional representation on the other hand seeks to provide representation to each and every individual in the society in the decisions and deliberations of the representative body.

Representativeness of an electoral outcome however has two distinct features, representativeness in decision making and representativeness in deliberations. For both the concepts of representativeness the ideal situation is direct democracy where every individual in the society gets to deliberate and decide on its fate. While representativeness in decision for all members of the society is desirable its practical implementation seems implausible as this would lead to every individual vying for his/her decision to prevail in the society. Therefore prevalence of the majority in decision making is more plausible. Nevertheless, less representation in decision making should not preclude representativeness in deliberations for all the members of the society. But there seems to lie a trade-off between this two notions of representativeness and exactly this trade-off forms the crux of the debate between majoritarianism and proportional representation. While majoritarian voting exaggerates the support of the majority in terms of inflated share of seats in the representative body, thereby weakening minority representation, on the other hand proportional representation systems lead to fragmentation of polity, coalition building and thereby giving unwarranted power in the hand of the minority to veto decisions of the majority. Both the systems of voting undermine the essence of representativeness which its counterpart upholds. While majoritarian principle violates representativeness in deliberations, proportional representation undermines representativeness in decision making.

Traditional social choice theory has paid scant attention to the debate arising out of the two differing notions of representativeness in the literature on voting. Hence exploring the issues related to representativeness of a preference aggregation process with more rigorous formalism can be a worthwhile venture.

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