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Numerical Methods For Certain Classes of Singular Two-Point Boundary Value Problems

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C E R T I F I C A T E

This work entitled "Numerical Methods For Certain Classes of Singular Two Point Boundary Value Problems" embodies in this dissertation has been carried out in the School of Computer and System Sciences, Jawaharlal Nehru University, New Delhi-110067 and is original and has not been submitted so far in part or full for any other degree or diploma of any University.

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C O N T E N T S

CHAPTER - I

A FOURTH-ORDER FINITE DIFFERENCE METHOD FOR A CLASS OF SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS

Abstract

We discuss the construction of finite difference approximations for the class of singular non linear two point boundary value problem :

$$(x^{\alpha} y')' = f(x,y), \quad y(0) = A, \quad y(1) = B,$$

$$0 < \alpha < 1.$$

We obtain a method of order four (for all $\alpha \in (0,1)$) involving three evaluations of f. For $\alpha = 0$ this method reduces to the Noumerov's method. Convergence of this method is established and illustrated by numerical examples.

1. Introduction

We consider the class of singular two point boundary value problem :

$$(1) \qquad (x^{\alpha} y')' = f(x,y), \qquad 0 < x \leq 1,$$

$$y(0) = A,$$

$$y(1) = B,$$

where α is a constant satisfying $0 < \alpha < 1$, and A, B are finite constants. We assume that for $(x,y) \in \{[0,1] \times R\}$,

$$(A) \quad \begin{cases} \text{(i)} & f(x,y) \text{ is continuous,} \\ \text{(ii)} & \frac{\partial f}{\partial y} \text{ exists and is continous and,} \\ \text{(iii)} & \frac{\partial f}{\partial y} \geq 0. \end{cases}$$

Certain classes of singular boundary value problems have been considered by Jamet [2, 3] and Parter [1] in the linear case only. Jamet studied the application of a standard three point finite difference scheme with a uniform mesh of size h and has shown that the error in the maximum norm is $O(h^{1-\alpha})$. Ciarlet et al [4] used a suitable Rayleigh - Ritz - Galerkin method and improved Jamet's result by showing that the error in the uniform norm for their Galerkin approximation is $O(h^{2-\alpha})$. Gusttafsson [5] gave a numerical method for solving singular boundary value problems by representing the solutions as a series expansion on a sub-interval near the singularity and by using difference methods for a regular boundary value problem derived for the remaining interval. Reddian [6] and Reddian and Schumaker [7] have studied collection for the solution of singular two point boundary value problems. Their methods concern projection into finite-dimensional linear spaces of singular non-polynomial splines, these singular splines possess convenient local support basis which have a certain advantage in numerical computations. Recently Chawla and Katti [8] have given a second-order method for (1), based on uniform mush.

In this chapter we present a fourth order finite difference method for the class of two point singular boundary value problem (1).

In section 2, using a certain identity based on uniform mesh over [0, 1], we obtain method of order four (for all $\alpha \in (0,1)$) based on three-evaluations of f. This method has the property that for $\alpha = 0$ it reduces to the well known Noumerov's method. $O(h^4)$ - convergence of this method is established under suitable conditions. Numerical illustrations are given in section 3, which establish $O(h^4)$ convergence of above method for various $\alpha \in (0,1)$.

2. The Finite Difference Method :

For a +ve integer $N \geq 2$, consider the uniform mesh over closed interval [0,1] : $0 = x_0 < x_1 < x_2 < \dots < x_N = 1$.

with $x_k = kh$. Let $y_k = y(x_k)$, $f_k = f(x_k, y_k)$ etc. Following Chawla and Katti [9], with $p(x) = x^{\alpha}$, we obtain the identity

$$(2) \qquad \frac{y_{k+1} - y_k}{J_k} - \frac{y_k - y_{k-1}}{J_{k-1}} = \frac{I_k^+}{J_k} + \frac{I_k^-}{J_{k-1}}, \quad k = 1(1)\,N-1$$

where we have set

$$(3) \qquad I_k^{\pm} = \frac{1}{(1-\alpha)} \int_{x_k}^{x_{k\pm1}} \left(x_{k\pm1}^{1-\alpha} - t^{1-\alpha} \right) f(t)\, dt$$

and

$$(4) \qquad J_k = \left(x_{k+1}^{1-\alpha} - x_k^{1-\alpha} \right) / \, (1-\alpha)$$

Using identity (2) various methods can be obtained for the singular two point boundary value problem (1). We are interested in obtaining method of order four based on three evaluations of f. In section 2.1 we obtain a method of order four based on uniform mesh and prove its convergence in section 2.3.

2.1 Fourth Order Method

We assume that

(5) $$\frac{I_k^+}{J_k} + \frac{I_k^-}{J_{k-1}} = C_{o,k}\, f_k + C_{1,k}\, f_{k+1} + C_{2,k}\, f_{k-1} + t_k\,(h).$$

where $c_{i,k}$'s are certain function of x_k's.

By Taylor expansion of f about x_k and comparing the coefficients of f, f' and f'' we find that

(6a) $$C_{o,k} = (-B_{2,k} + h^2 B_{o,k}) / h^2$$

(6b) $$C_{1,k} = (B_{2,k} + h B_{1,k}) / 2h^2$$

(6c) $$C_{2,k} = (B_{2,k} - h B_{1,k}) / 2h^2$$

where

(7) $$B_{m,k} = \sum_{f=o}^{m+1} \left[\sum_{i=-1}^{0} \frac{(-1)^{i+j+2}}{J_{k+i}} \left(x_{k+1+i}^{m+2-\alpha-j} - x_{k+i}^{m+2-\alpha-j}\right) \right] \frac{x_k^{j}}{(m+2-\alpha-j)} \binom{m+1}{j}$$

Then

(8) $$t_k\,(h) = C_{3,k}\, f_k''' + \frac{1}{6} \int_{x_{k-1}}^{x_{k+1}} G(s)\, f^{(4)}(s)\,ds$$

where

(9) $$C_{3,k} = (B_{3,k} - h^2 B_{1,k})/6$$

(10) $$G(s)=\begin{cases} \frac{1}{4J_k}\sum_{j=0}^{4}(-1)^j \binom{4}{j}\frac{s^j}{(5-\alpha-j)}\left(x_{k+1}^{5-\alpha-j} - s^{5-\alpha-j}\right), & x_k \leqslant s \leqslant x_{k+1} \\ \frac{1}{4J_{k-1}}\sum_{j=0}^{4}(-1)^j \binom{4}{j}\frac{s^j}{(5-\alpha-j)}\left(s^{5-\alpha-j} - x_{k-1}^{5-\alpha-j}\right), & x_{k-1} \leqslant s \leqslant x_k \end{cases}$$

with the help of (5)and (8) we obtain

(11) $$-\frac{1}{J_{k-1}}y_{k-1} + \left(\frac{1}{J_k} + \frac{1}{J_{k-1}}\right)y_k - \frac{1}{J_k}y_{k+1} + C_{o,k}\, f_k +$$

$$C_{1,k}\, f_{k+1} + C_{2,k}\, f_{k-1} + t_k(h) = 0, \quad K= 1(1)N-1$$

A finite difference method can now be based on the discretization (11) of the differential equation together with the boundary conditions; note that each discretization in (11) is based on three evaluation of f. Our method can now be based on (11) neglecting $t_k(h)$; the fourth order convergence of this method is given in Sec. 2.3

2.2 Matrix Formulation of our Finite Difference Method

It is convenient to describe the above method in matrix form. Let $D = (d_{i,j})_{i,j=1}^{N-1}$ denote the tridiagonal matrix with

$$d_{k,k} = \frac{1}{J_k} + \frac{1}{J_{k-1}}, \qquad k = 1(1)N-1,$$

$$d_{k,k+1} = -\frac{1}{J_k}, \qquad k = 1(1)N-2,$$

$$d_{k,k-1} = -\frac{1}{J_{k-1}}, \qquad k = 2(1)N-1,$$

let

$$P = (p_{ij})_{i,j=1}^{N-1} \quad \text{denote the tridiagonal matrix}$$

with

$$p_{k,k} = C_{o,k}, \qquad k = 1(1)N-1$$

$$p_{k,k+1} = C_{1,k}, \qquad k = 1(1)N-2$$

$$p_{k,k-1} = C_{2,k}, \qquad k = 2(1)N-1,$$

and let

$$Q = (q_1, 0, \ldots \ldots \ldots \ldots \ldots q_{N-1})^T$$

where

$$q_1 = - C_{2,1} f_o + \frac{A}{J_o} ,$$

$$q_{N-1} = - C_{1,N-1} f_N + \frac{B}{J_{N-1}}.$$

Also, let

$$Y = (y_1, y_2, \dots \dots \dots , y_{N-1})^T$$

$$F(Y) = (f_1, f_2, \dots \dots \dots, f_{N-1})^T$$

$$\text{and } T = (t_1, t_2, \dots \dots \dots, t_{N-1})^T$$

Then the finite difference discretization described by (11) can be expressed in the matrix form as

(12) $DY + PF(Y) + T = Q$

Our method now consists of finding an approximation $\widetilde{Y}$ for Y by solving the (N-1) x (N-1) system :

(13) $D\widetilde{Y} + PF(\widetilde{Y}) = Q$

In case $f(x,y)$ is linear, (13) leads to a tridiagonal linear system; in the non-linear case the system (13) can be solved by Newton- Raphson method and an adaptation of Gauss elimination for tridiagonal linear system.

2.3 Convergence of the Method

We next show that the method described by (13) is $O(h^4)$ - convergent for all $\alpha \in (0,1)$.

let

$$E = (e_1, \ldots\ldots, e_{N-1})^T$$
$$= \widetilde{Y} - Y$$

we may write

(14) $f(x_k, \widetilde{y}_k) - f(x_k, y_k) = e_k U_k, \quad k = 1(1)N-1$

for suitable U_k's ; note that $U_k \geqslant 0$.

With the help of (14) from (12) and (13) we obtain the error equation

(15) $(D + PM)\, E = T$

where

$$M = \text{diag} \left\{ U_1, \ldots\ldots U_{N-1} \right\}$$

It is easy to see that, for sufficiently small h, D + PM is irreducible and monstone and PM $\geqslant 0$. Therefore $(D + PM)^{-1}$ exists.

$(D + PM)^{-1} \geqslant 0$ and

(16) $(D + PM)^{-1} \leqslant D^{-1}$

So from (15) and (16) we have

(17) $\| E \| \leqslant \| D^{-1} T \|$

Using the usual arguments for inverting a symmetric tridiagonal matrix, it can be shown that (see Appendix [1]).

if $D^{-1} = (d^{-1}_{i,j})$, then,

(18) $d^{-1}_{i,j} = x_i^{1-\alpha}(1-x_j^{1-\alpha}) / (1-\alpha), \quad i \leqslant j$

$= x_j^{1-\alpha}(1-x_i^{1-\alpha}) / (1-\alpha), \quad i \geqslant j$

We next obtain bounds for the local truncation error t_k. For sufficiently small h, we see that G(s) has the same sign in (x_{k-1}, x_{k+1}). Hence (8)' can be written as

(19) $t_k = C_{3,k} f_k''' + C_{4,k} f^{(4)}(\sigma_k)$

where

$$C_{4,k} = (B_{4,k} - C_{1,k} h^4 - C_{2,k} h^4)/24$$

Since for fixed x_k

$$(20) \qquad \lim_{h \to 0} \frac{C_{3,k}}{h^5} = - \frac{\alpha \, x_k^{5\alpha - 1}}{24(1-\alpha)^5}, \qquad k = 1(1)N-1,$$

and

$$(21) \qquad \lim_{h \to 0} \frac{C_{4,k}}{h^5} = - \frac{x_k^{5\alpha}}{240 \, (1-\alpha)^5}, \qquad k = 1(1)N-1$$

It follows that for sufficiently small h,

$$(22) \qquad |C_{3,k}| < \frac{\alpha \, x_k^{5\alpha - 1}}{12 \, (1-\alpha)^5} \, h^5$$

and

$$(23) \qquad |C_{4,k}| < \frac{x_k^{5\alpha}}{120(1-\alpha)^5} \, h^5 .$$

We assume that

$$x^{3\alpha} \, |f'''| \leqslant N_1 \, ,$$

$$(24) \qquad x^{3\alpha + 1} \, |f^{(4)}| \leqslant N_2 \, , \qquad 0 < x \leqslant 1,$$

for suitable positive constants N_1 and N_2. Then with the help of (22) and (23) from (19) we obtain

(25) $$|t_k| \leqslant Ch^5 \; x_k^{2\alpha-1}$$

where

$$C = \frac{10\alpha N_1 + N_2}{120\,(1-\alpha)^5}$$

Using (18) and (25), from (17) we obtain

(26) $$|e_i| \leqslant \sum_{j=1}^{N-1} d_{i,j}^{-1} \; |t_j| \quad , \quad i = 1(1)\, N-1$$

$$\leqslant \frac{Ch^5}{(1-\alpha)} \left[(1-x_i^{1-\alpha}) \sum_{j=1}^{i} x_j^{\alpha} + x_i^{1-\alpha} \sum_{j=i+1}^{N-1} (x_j^{2\alpha-1} - x_j^{\alpha}) \right].$$

It is easy to establish the inequality:

(27) $$h \sum_{j=1}^{i} x_j^{\alpha} < \int_0^{x_i} x^{\alpha}\, dx = \frac{x_i^{1+\alpha}}{(1+\alpha)}$$

Again,

$$h \sum_{j=i+1}^{N-1} (x_j^{2\alpha-1} - x_j^{\alpha}) < \int_{x_i}^{x_N} (x^{2\alpha-1} - x^{\alpha})\, dx$$

(28) $$< \frac{x_N^{2\alpha} - x_i^{2\alpha}}{2\alpha} - \frac{x_N^{1+\alpha} - x_i^{1+\alpha}}{1+\alpha}$$

With the help of (27) and (28) together with $X_N = 1$, from (26) we obtain

(29) $$|e_i| \leqslant \frac{Ch^4}{2\alpha(1+\alpha)} x_i^{1-\alpha}(1 - x_i^{2\alpha})$$

It can now be shown that for $i = 1(1)\,N-1$,

(30) $$x_i^{1-\alpha}(1 - x_i^{2\alpha}) < 1$$

With the help of (30) from (28) we obtain for sufficiently small h,

(31) $$\|E\| = \max_{1 \leqslant i \leqslant N-1} |e_i|$$

$$= C^* h^4 , \qquad C^* = \frac{C}{2\alpha(1+\alpha)}$$

We have thus established the following result :

Theorem :

Assume that f satisfies (A); further let

$f^{(4)} \in C\{[0,1] \times R\}$, $x^{3\alpha}|f'''|$,

$x^{3\alpha+1}|f^{(4)}| \in C\{[0,1] \times R\}$.

Then for the method based on (11) with $x_k = kh$, we have for sufficiently small h, for all $\alpha \in (0,1)$,

(32) $\quad \|E\| = O(h^4)$

3. Numerical Illustrations :

We next illustrate our method by considering the following three examples.

Example 1

We consider the non linear differential equation

$$(x^{\alpha} y')' = \beta x^{\alpha+\beta-2} (\beta x^{\beta} e^{y} - (\alpha+\beta-1))/(4 + x^{\beta})$$

subject to boundary conditions

$$y(0) = \ln\left(\frac{1}{4}\right) \text{ and } y(1) = \ln\left(\frac{1}{5}\right)$$

With exact solution $y(x) = \ln(1/(4+x^{\beta}))$

For $N = 2^k$, $k = 3(1)8$, the corresponding values of $\|E\|$ are shown in table 1.

TABLE - 1

N	$\|\| E \|\|$
α = 0.25,	β = 4.0
8	4.1 (-5)
16	2.5 (-6)
32	1.6 (-7)
64	9.9 (-8)
128	6.2 (-10)
256	3.9 (-11)
α = 0.5	β = 3.0
8	7.6 (-5)
16	4.7 (-6)
32	2.9 (-7)
64	1.8 (-8)
128	1.4 (-9)
256	7.2 (-11)
α = 0.8	β = 1.8
8	6.9 (-4)
16	4.6 (-5)
32	2.9 (-6)
64	1.9 (-7)
128	1.2 (-8)
256	7.3 (-10)

Example 2

We consider the linear differential equation

$$(x^{\alpha} y')' = \beta x^{\alpha+\beta-2} ((\alpha+\beta-1)+\beta x^{\beta})y$$

subject to boundary conditions

$y(0) = 1$ and $y(1) = e$

with the exact solution $y(x) = \exp(x^{\beta})$

For $N = 2^k$, $k = 3(1)8$, the corresponding value of $\|E\|$ are shown in table 2.

TABLE - 2

N	$\|E\|$
	$\alpha = 0.25$, $\beta = 4.0$
8	9.3 (-3)
16	6.4 (-4)
32	4.1 (-5)
64	2.6 (-6)
128	1.6 (-7)
256	1.0 (-8)

$\alpha = 0.5$,	$\beta = 3.0$
8	1.4 (-2)
16	1.0 (-3)
32	6.6 (-5)
64	4.2 (-6)
128	2.6 (-7)
256	1.6 (-8)

$\alpha = 0.8$,	$\beta = 1.8$
8	3.7 (-2)
16	3.6 (-3)
32	2.6 (-4)
64	1.7 (-5)
128	1.1 (-6)
256	6.6 (-8)

Example 3

We consider the linear differential equation

$$(x^{\alpha} y')' = -(x \text{ Cos }(x) + (2-\alpha) \text{ Sin}(x))$$

subject to the boundary conditions

$$y(0) = 0 \ , \quad y(1) = \text{Cos } 1$$

with the exact solution $y(x) = x^{1-\alpha} \text{ Cos } x$.

This example has been considered by Gusttafsson [5].

For $N = 2^k$, $k = 3(1)8$, the corresponding value of $\|E\|$ are shown in table 3.

TABLE - 3

N	$\|E\|$
$\alpha = 0.25$	
8	4.7 (-6)
16	2.9 (-7)
32	1.8 (-8)
64	1.1 (-9)
128	7.1 (-11)
256	4.2 (-12)
$\alpha = 0.5$	
8	2.7 (-5)
16	1.7 (-6)
32	1.1 (-7)
64	6.6 (-9)
128	4.1 (-10)
256	2.5 (-11)

$\alpha = 0.8$

8	8.5 (-4)
16	6.1 (-5)
32	3.9 (-6)
64	2.5 (-7)
128	1.5 (-8)
256	9.6 (-10)

CHAPTER - II

A NEW FINITE DIFFERENCE METHOD AND ITS CONVERGENCE FOR A CLASS OF SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS

PART -I

Abstract

A new finite difference method based on uniform mesh is given for the (weakly) singular two point boundary value problems:

$$x^{\alpha} y'' = f(x,y), \quad y(0) = A, \; y(1) = B, \; 0 < \alpha < 1.$$

Under quite general conditions on f' and f'', we show that our method based on uniform mesh provides $O(h^2)$ convergent approximations for all $\alpha \in (0,1)$. Our method is based on one evaluation of f and for $\alpha = 0$ it reduces to the classical second order method for $y'' = f(x,y)$.

1. Introduction

Consider the (weakly) singular two point boundary value problem :

$$(1) \qquad x^{\alpha} y'' = f(x,y), \qquad 0 < x \leqslant 1$$
$$y(0) = A, \quad y(1) = B.$$

Here $\alpha \in (0,1)$ and A, B are finite constants. We assume that for $(x,y) \in \{[0,1] \times R\}$

(A)
(i) $f(x,y)$ is continuous

(ii) $\frac{\partial f}{\partial y}$ exists and is continuous

(iii) $\frac{\partial f}{\partial y} \geqslant 0$

The above problem occurs in various branches of engineering, mechanies etc. Such a problem has extensively been dealt with by Mayers [10]. The purpose here is to give a simple finite difference method based on uniform mesh for the singular two point boundary value problem (1). The method is based on one - evaluation of f. Under quite general conditions on f' and f'' we show that our present method provides $O(h^2)$ - convergent approximations for all $\alpha \in (0,1)$. The present method, its second order convergence for various $\alpha \in (0,1)$ and the conditions guaranteering convergence are illustrated by an example.

2. The Finite Difference Method

For a + ve integer $N \geqslant 2$, consider the uniform mesh over closed interval $[0,1]$: $x_k = kh$, $k = 0(1)N$, $h = \frac{1}{N}$. Let $y_k = y(x_k)$, $f_k = (x_k, y_k)$ etc.

We write (1) in the form

(2) $y'' = x^{-\alpha} f(x,y)$

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we set

$$z(x) = y' ;$$

Integrating (2) from x_k to x, we obtain

$$(3) \qquad z(x) = z_k + \int_{x_k}^{x} t^{-\alpha} f(t)\, dt$$

where

$$f(t) = f(t, y(t)).$$

Integrating (3) from x_k to x_{k+1} and interchanging the order of integration, we obtain

$$(4) \qquad y_{k+1} - y_k = z_k . h + \int_{x_k}^{x_{k+1}} (x_{k+1} - t) t^{-\alpha} f(t)\, dt.$$

Similarly

$$(5) \qquad y_k - y_{k-1} = z_k . h - \int_{x_{k-1}}^{x_k} (t - x_{k-1}) t^{-\alpha} f(t)\, dt.$$

Eliminating $z_{k:h}$ from (4) and (5) we obtain the identity :

$$(6) \qquad y_{k+1} - 2y_k + y_{k-1} = \int_{x_k}^{x_{k+1}} (x_{k+1} - t) t^{-\alpha} f(t) dt + \int_{x_{k-1}}^{x_k} (t - x_{k-1}) t^{-\alpha} f(t)\, dt ,$$

$$k = 1(1)\, N-1$$

Identity (6) is our basic result from which methods of various orders can be obtaiied for the two point boundary value problem (1). However, we shall be interested here in obtaining a method of order two which will be based on 1 evaluation of f.

By Taylor's expansion of f about x_k, we obtain

(7) $$f(t) = f_k + (t-x_k)\, f'_k + \frac{1}{2!}(t-x_k)^2\, f''(\xi_k).$$

where $\xi_k \in (x_{k-1}, x_{k+1})$

with the help of (7) from (6), we obtain

(8) $$-y_{k-1} + 2y_k - y_{k+1} + B_{o,k}\, f_k + t_k = 0,$$

$k = 1(1)\ N-1$

where

$$B_{o,k} = \frac{1}{(1-\alpha)(2-\alpha)}\left(x_{k-1}^{2-\alpha} - 2x_k^{2-\alpha} + x_{k+1}^{2-\alpha} \right)$$

and

(9) $$t_k = B_{1,k}\, f'_k + \frac{1}{2}\, B_{2,k}\, f''(\xi_k),$$

$$\xi_k \in (x_{k-1}, x_{k+1}).$$

where

$$B_{m,k} = \sum_{j=1}^{m+1} \frac{(-1)^{j+1}\, C\, x_k^{\,j-1}}{(m+3-\alpha-j)(m+2-\alpha-j)}$$

$$\left(x_{k+1}^{\,m+3-\alpha-j} - 2x_k^{\,m+3-\alpha-j} + x_{k-1}^{\,m+3-\alpha-j} \right),$$

$$m = 1, 2$$

and

$$C = \begin{cases} 1 & \text{for all } m \,\&\, j \\ 2 & \text{for } m = 2,\ j = 2 \end{cases}$$

A finite difference method can now be based on the discretization (8) of differential equation involving one evaluation of f. In section 4, we show that, under suitable conditions, our method based on (8) is $O(h^2)$-convergent.

3. Matrix Formulation of our Finite Difference Method

It is convenient to express the above discretization in matrix form. Let $D = (d_{i,j})_{i,j=1}^{N-1}$ denote the tridiagonal matrix with

$$d_{k,k-1} = -1, \qquad k = 2(1)\,N-1$$

(10) $$d_{k,k} = 2, \qquad k = 1(1)\,N-1$$

$$d_{k,k+1} = -1, \qquad k = 1(1)\,N-2,$$

and

$P = (p_{ij})$ denote the diagonal matrix with

$p_{k,k} = B_{o,k}$, $k = 1(1)\ N-1$

Let

$Q = (q_1, 0 .. .. 0, q_{N-1})^T$

where

$q_1 = A$

$q_{N-1} = B,$

Also, let

$F(y) = (f_1, ... f_{N-1})^T$,

$Y = (y_1, ..., y_{N-1})^T$,

and $T = (t_1, ... , t_{N-1})^T$

Thus the discretization (8) together with the boundary conditions can be expressed as :

(11) $DY + PF(Y) + T = Q$

and a method based on (8) consists of finding an approximation $\widetilde{Y}$ for Y by solving the $(N-1)x(N-1)$ system :

(12) $\quad D\tilde{Y} + PF(\tilde{Y}) = Q$

In case the differential equation is linear in y, (12) is tridiagonal linear system; in the case of non linear differential equation, the non-linear system can be solved by Newton - Raphson method and an adaptation of Gauss - elimination for tridiagonal linear systems.

4. Convergence of the Method

We next establish convergence of our finite difference method showing that under suitable conditions the above method provides $O(h^2)$ - convergent approximations for all $\alpha \in (0,1)$.,

let

$$E = (e_1, \ldots \ldots, e_{N-1})^T$$
$$= \tilde{Y} - Y$$

we may write

(13) $\quad f(x_k, \tilde{y}_k) - f(x_k, y_k) = e_k \; U_k, \quad k = 1(1)\, N-1$

for suitable U_k's. Now

(14) $F(Y) - F(Y) = ME$

where

$$M = (m_{ij})_{i,j=1}^{N-1}$$ is the diagonal matrix with

(15) $m_{k,k} = U_k, \quad k = 1(1)\ N-1$

(note that $U_k \geqslant 0$)

With the help of (14), from (11) and (12) we obtain the error equation:

(16) $(D + PM)\ E = T$

To show that our method is $O(h^2)$ - convergent we first establish the following lemmas.

Lemma 1 :- $B_{o,k} > 0$ for $k = 1(1)\ N-1$

Proof :

let

(17) $f(x) = x^{2-\alpha} - (x-1)^{2-\alpha}$,

(18) $f'(x) = (2-\alpha)\left[x^{1-\alpha} - (x-1)^{1-\alpha}\right]$

$> 0 \quad \text{for} \quad x \geqslant 1$

So, f(x) is strictly increasing function of x which gives,

$$f(x+1) > f(x)$$

$$f(x+1) - f(x) > 0$$

$$(k+1)^{2-\alpha} - 2k^{2-\alpha} + (k-1)^{2-\alpha} > 0.$$

This completes the proof of lemma 1.

Lemma 2 :- The inverse of the matrix D is given as

$$d^{-1}_{i,j} = \frac{i(N-j)}{N} , \quad i \leqslant j \tag{19}$$

$$= \frac{j(N-i)}{N} , \quad i \geqslant j$$

Proof :

The proof is as given in Jain [11]

Before doing the convergence analysis we mention the following results

let $W = \{1, 2, \ldots\ldots\ldots, n\}$

<u>Definition</u> 1 :

A matrix $A = (a_{i,j})$ of order $n \geqslant 2$ is irreducible if for any two integer i and j, $i \in w$, $j \in w$, there exist a sequence of non-zero elements of A of the form

$$\left\{a_{ii_1}, a_{i_1i_2}, \ldots \ldots \ldots, a_{i_{n-1}j}\right\}$$

Theorem 1 :

A tridiagonal matrix $A = (a_{ij})$ is irreducible if and only if

$a_{i,i-1} \neq 0 \ (i = 2,3,\ldots, n)$ and

$a_{i,i+1} \neq 0 \ (i = 1,2,\ldots,n-1)$

Definition 2 :

A matrix A with real elements is called monotone if $AZ \geqslant 0$ implies $Z \geqslant 0$

Theorem 2 :

A matrix A is monotone if and only if the elements of inverse matrix A^{-1} are non-negetive.

Theorem 3 :

Let the matrix $A = (a_{i,j})$ be irreducible and satisfy the conditions,

(i) $a_{i,j} \leqslant 0$, $i \neq j$; $i, j = 1, \ldots n$

(ii) $\sum_{j=1}^{n} a_{i,j} \begin{cases} \geqslant 0 & i = 1,2,\ldots\ldots n \\ > 0 & \text{for at least one } i \end{cases}$

The proofs of theorem 1,2 and 3 are given in Henriei [12].

Since for all h, D and D + PM are irreducible and monotone and $(D + PM) \geqslant D$

we have

$$(D + PM)^{-1} \leqslant D^{-1}$$

From (16) we obtain

(20) $$\|E\| \leqslant \|D^{-1} T\|$$

We next obtain a bound on the local truncation error. Since for fixed x_k,

$$\lim_{h \to 0} \frac{B_{1,k}}{h^4} = -\frac{\alpha}{6} x_k^{-1-\alpha}$$

and

$$\lim_{h \to 0} \frac{B_{2,k}}{h^4} = \frac{1}{6} x_k^{-\alpha}$$

it follows that for sufficiently small h,

(21) $$|B_{1,k}| < \frac{\alpha}{3} x_k^{-1-\alpha} . h^4 , \quad k = 1(1) N-1$$

(22) $\quad |B_{2,k}| < \frac{1}{3} x_k^{-\alpha} . h^4 \quad , \quad k = 1\ (1)\ N-1$

Now let α be fixed in (0,1) and let β be chosen such that $\alpha + \beta < 1$

we assume that

(23) $\quad x^{\beta} |f'| \leqslant N_1$

(24) $\quad x^{1+\beta} | f''| \leqslant N_2, \quad 0 < x \leqslant 1.$

N_1 and N_2 are suitable positive constants. With the help of (21), (22) and (23), (24) from (9) we obtain for sufficiently small h,

(25) $\quad |t_k| \leqslant Ch^4 \quad x_k^{-1-(\alpha + \beta)}$

where

$$C = \frac{2\alpha N_1 + N_2}{6}$$

Now with the help of (25) from (20) we obtain

$$|e_i| \leqslant \sum_{j=1}^{N-1} d_{ij}^{-1} \cdot |t_j|$$

$$\leqslant Ch^{4-(1+\alpha+\beta)} \left[\sum_{j=1}^{i} d_{ij}^{-1} \ j^{-1-(\alpha+\beta)} + \sum_{j=i+1}^{N-1} d_{ij}^{-1} \ j^{-1-(\alpha+\beta)} \right]$$

$$\leq Ch^{3-(\alpha+\beta)} \Big[(N-i) \sum_{j=1}^{i} \frac{j.\ j^{-1-(\alpha+\beta)}}{N}$$

$$+ i \sum_{j=i+1}^{N-1} \frac{(N-j)}{N} j^{-1-(\alpha+\beta)} \Big]$$

$$\leq \frac{Ch^{3-(\alpha+\beta)}}{N} \Big[(N-i) \int_0^i j^{-(\alpha+\beta)} d_j$$

$$+ i \int_i^N (Nj^{-1-(\alpha+\beta)} - j^{-(\alpha+\beta)}) dj \Big]$$

$$\leq \frac{Ch^{4-(\alpha+\beta)} \left[Ni^{1-\alpha-\beta} - iN^{1-\alpha-\beta} \right]}{(\alpha+\beta)(1-\alpha-\beta)} \quad i = 1(1)\ N-1$$

Let us now consider the following f(x) as a continuous function of $x \in [1,\ N-1]$

$$f(x) = Nx^{1-\alpha-\beta} - xN^{1-\alpha-\beta}$$

For maximum of f(x)

$$f'(x) = N(1-\alpha-\beta)\ x^{-\alpha-\beta} - N^{1-\alpha-\beta} = 0$$

gives

$$i = x = (1-\alpha-\beta)^{\frac{1}{\alpha+\beta}} .\ N$$

$$\|E\| = \max_{1 \leqslant i \leqslant N-1} (e_i)$$

$$= \frac{Ch^{4-\alpha-\beta}}{(\alpha+\beta)(1-\alpha-\beta)} \left[N.(1-\alpha-\beta)^{\frac{(1-\alpha-\beta)}{(\alpha+\beta)}} . N^{1-\alpha-\beta} . -(1-\alpha-\beta)^{\frac{1}{\alpha+\beta}} N.N^{1-\alpha-\beta} \right]$$

$$= \frac{Ch^{4-\alpha-\beta} N^{2-\alpha-\beta}}{(\alpha+\beta)(1-\alpha-\beta)} (1-\alpha-\beta)^{\frac{1}{\alpha+\beta}} . \left[\frac{1}{1-\alpha-\beta} - 1 \right]$$

$$= C^{*} h^{2}$$

where $C^{*} = \dfrac{C}{(1-\alpha-\beta)^{\left(2-\frac{1}{\alpha+\beta}\right)}}$

5. Numerical Illustration

To illustrate our method and its $O(h^2)$ - convergence we consider the following example.

Example

We consider the linear differential equation

$$x^{\alpha} y'' = (2\beta - 1)\, x^{\alpha+\beta-2} + \beta(\beta-1) x^{\alpha+\beta-2} \log x$$

subject to the boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(1) = 0$$

exact solution is

$$y = x^{\beta} \log x$$

For $N = 2^k$, $k = 2(1)6$, the corresponding value of $||E||$ are shown in table.

TABLE

N	$\|\|E\|\|$
$\alpha = 0.25$,	$\beta = 3.50$
4	1.7 (-2)
8	4.2 (-3)
16	1.0 (-3)
32	2.6 (-4)
64	6.5 (-5)

N	$\|\|E\|\|$
$\alpha = 0.50$, $\beta = 3.00$	
4	1.3 (-2)
8	3.9 (-3)
16	1.1 (-3)
32	2.7 (-4)
64	6.9 (-5)
$\alpha = 0.50$, $\beta = 1.50$	
4	2.5 (-2)
8	1.1 (-2)
16	4.5 (-3)
32	1.8 (-3)
64	6.8 (-4)
$\alpha = 0.75$, $\beta = 2.75$	
4	1.2 (-2)
8	4.9 (-3)
16	1.6 (-3)
32	4.4 (-4)
64	1.2 (-4)

N	$\|\|E\|\|$
$\alpha = 0.80,$	$\beta = 3.00$
4	1.7 (-2)
8	5.6 (-3)
16	1.5 (-3)
32	3.9 (-4)
64	9.7 (-5)
$\alpha = 0.99,$	$\beta = 3.00$
4	2.2 (-2)
8	7.3 (-3)
16	2.0 (-3)
32	4.9 (-4)
64	1.2 (-4)

PART -II

1. Introduction

In Part I of this chapter we have dealt the problem (1) for $0 < \alpha < 1$. Here we are extending the same problem for $\alpha = 1$, with the same boundary conditions. So our (weakly) singular two point boundary value problem becomes

$$(1) \qquad xy'' = f(x,y), \qquad 0 < x \leqslant 1$$

$$y(0) = A, \qquad y(1) = B$$

Our method is based on one evaluation of f, under quite general conditions on f' and f''. We show that this method provides $O(h \log h)^2$ - convergent.

2. The Finite Difference Method

For a + ve integer $N \geqslant 2$, consider the uniform mesh over closed interval $[0,1]$: $0 = x_0 < x_1 < x_2 \dots$ $\dots, < x_N = 1$, with $x_k = kh$.

Let

$$y_k = y(x_k), \quad f_k = f(x_k, y_k) \text{ etc.}$$

We write (1) in the form

$$(2) \quad y'' = x^{-1} \cdot f(x,y)$$

we set

$$z(x) = y'$$

Integrating (2) from x_k to x, we obtain

$$(3) \quad z(x) = z_k + \int_{x_k}^{x} t^{-1} f(t)\, dt$$

where

$$f(t) = f(t, y(t))$$

Integrating (3) from x_k to x_{k+1} and interchanging the order of integration, we obtain

$$(4) \quad y_{k+1} - y_k = z_k.h + \int_{x_k}^{x_{k+1}} (x_{k+1} - t)\, t^{-1} f(t)dt$$

In an analogus manner, we obtain

$$(5) \quad y_k - y_{k-1} = z_k.h - \int_{x_{k-1}}^{x_k} (t - x_{k-1})t^{-1} f(t)dt$$

Eliminating $z_k.h$ from (4) and (5). We obtain the identity :

(6) $y_{k+1} - 2y_k + y_{k-1}$

$$= \int_{x_k}^{x_{k+1}} (x_{k+1}-t)t^{-1} \; f(t)dt + \int_{x_{k-1}}^{x_k} (t-x_{k-1})t^{-1} \; f(t)dt,$$

$k = 1(1)N-1$

Identity (6) is our basic result from which methods of various orders can be obtained for the two point boundary value problem (1). However, we shall be interested here in obtaining a method, which will be based on 1 evaluation of f.

By Taylor's expansion of f about x_k, we obtain

(7) $$f(t) = f_k + (t - x_k) \; f'_k + \frac{1}{2!} (t - x_k)^2 f''(\xi_k)$$

where $\xi_k \in (x_{k-1}, x_{k+1})$

with the help of (7) from (6), we obtain

(8) $$- y_{k-1} + 2y_k - y_{k+1} + B_{o,k} f_k + t_k = 0$$

$k = 2(1) N-1$

where

$$B_{o,k} = x_{k+1} \log \frac{x_{k+1}}{x_k} - x_{k-1} \log \frac{x_k}{x_{k-1}}$$

$$= x_{k+1} \log x_{k+1} - 2x_k \log x_k + x_{k-1} \log x_{k-1}$$

and

(9) $$t_k = B_{1,k} f_k' + \frac{1}{2} B_{2,k} f''(\xi_k),$$

$$\xi_k \in (x_{k-1}, x_{k+1})$$

where

$$B_{m,k} = (-1)^{m+1} x_k^m \left(x_{k-1} \log \frac{x_k}{x_{k-1}} - x_{k+1} \log \frac{x_{k+1}}{x_k} \right)$$

$$+ \sum_{j=0}^{m-1} (-2x_k)^j \cdot \left(\frac{x_{k+1}^{m+1-j} - 2x_k^{m+1-j} + x_{k-1}^{m+1-j}}{(m+1-j)!} \right),$$

$$m = 1,2$$

We note that the discretization (8) for differential equation (1) holds for $k = 2(1)\,N-1$. For obtaining the discretization corresponding to $k = 1$ we proceed as follows:

putting $k = 1$ in (6) we obtain

(10) $$y_2 - 2y_1 = \int_{x_1}^{x_2} \left(\frac{x_2}{t} - 1 \right) f(t)\, dt + \int_{x_o}^{x_1} f(t)\, dt$$

Since $y_o = A$

and $x_o = 0$

From (10) we obtain

(11) $$-y_2 + 2y_1 + B_{o,1} f_1 + t_1 = A$$

where $B_{o,1} = x_2 \log \frac{x_2}{x_1}$

(12) and

$$t_1 = B_{1,1} f_1' + \frac{1}{2} B_{2,1} f''(\xi_1), \quad x_1 < \xi_1 < x_2$$

and

$$B_{1,1} = -\frac{7}{12} h^2$$

$$B_{2,1} = \frac{2}{3} h^3$$

A finite difference method can now be based on the discretizations (8) and (11) of differential equation involving one evaluation of f. In section 4 we show that, under suitable conditions, our method based on (8) and (11) is $O(h \log h)^2$ convergent.

3. Matrix Formulation of Our Finite Difference Method

It is convenient to express the above discretization in matrix form. Let $D = (d_{i,j})_{i,j=1}^{N-1}$ denote the tridiagonal matrix with

$$(13) \qquad d_{k,k-1} = -1 \qquad k = 2(1)\,N-1$$

$$d_{k,k} = 2 \qquad k = 1(1)\,N-1$$

$$d_{k,k+1} = -1 \qquad k = 1(1)\,N-2$$

let

$P = (p_{ij})_{i,j=1}^{N-1}$ denote the diagonal matrix with

$$p_{1,1} = B_{o,1}$$

$$p_{k,k} = B_{o,k}\,, \qquad k = 2(1)\,N-1$$

let

$$Q = (q_1,\ 0,\ \ldots\ldots\ 0,\ q_{N-1})^T$$

$$q_1 = A$$

$$q_{N-1} = B$$

Also, let

$$F(y) = (f_1, \ldots \ldots \ldots, f_{N-1})^T$$

$$Y = (y_1, \ldots \ldots \ldots, y_{N-1})^T$$

and $$T = (t_1, \ldots \ldots \ldots, t_{N-1})^T$$

Thus the discretizations (8) and (11) can be expressed in matrix form :

(14) $$DY + PF(Y) + T = Q$$

and a method based on (8) consists of finding an approximation $\widetilde{Y}$ for Y by solving the $(N-1)x(N-1)$ system :

(15) $$D\widetilde{Y} + PF(\widetilde{Y}) = Q$$

In case the differential equation is linear in $\ddot{y}$, (12) is tridiagonal linear system; in the case of non-linear differential equation, the non-linear system can be solved by Newton - Raphson method and an adaptation of Gauss - elimination for tridiagonal linear systems.

4. Convergence of the Method

We next establish convergence of our finite difference method showing that under suitable conditions the above method provides $O((h \log h)^2)$ - convergent approximation for $\alpha = 1$ and the uniform mesh $x_k = kh$.

let

$$E = (e_1, \ldots\ldots, e_{N-1})^T$$

$$= \widetilde{Y} - Y$$

we may write

(16) $f(x_k, \widetilde{y_k}) - f(x_k, y_k) = e_k \; U_k , \quad k = 1(1)N-1$

for suitable U_k, now

(17) $F(\widetilde{Y}) - F(Y) = ME$

where

$M = (m_{i,j})_{i,j=1}^{N-1}$ is a diagonal matrix with

$m_{k,k} = U_k , \qquad k = 1(1)\ N-1$

(note that $U_k \geqslant 0$)

with the help of (17), from (14) and (15) we obtain the error equation :

(18) $(D + PM)\, E = T$

To show that our method is $O(h \log h)^2$ convergent we first establish the following lemmas

Lemma 1 :

$$B_{o,1} > 0$$

Proof :

$$B_{o,1} = x_2 \log \frac{x_2}{x_1}$$
$$> 0$$

this complete the proof of lemma.

Lemma 2 :

$$B_{o,k} > 0 , \qquad k = 2(1)\, N-1$$

Proof :

$$B_{o,k} = x_{k+1} \log x_{k+1} - 2x_k \log x_k + x_{k-1} \log x_{k-1}$$

let

$$f(k) = kh\log kh - (k-1)h \log (k-1)h$$

$$f(k) = kh \log k - (k-1)h \log(k-1) + h \log h$$

$$f(x) = xh \log x - (x-1)h \log (x-1) + h \log h$$

$$f'(x) = h \left[\log x - \log (x-1) \right] , \quad x \geqslant 1$$
$$> 0$$

So, f(x) is strictly increasing function of x which gives

$$f(x+1) > f(x)$$

$$f(x+1) - f(x) > 0$$

$$x_{k+1} \log x_{k+1} - 2x_k \log x_k + x_{k-1} \log x_{k-1} > 0$$

This completes the proof of lemma 2.

Lemma 3 :

The inverse of the matrix D is given by

$$(19) \quad d^{-1}_{i,j} = \frac{i\,(N-j)}{N}, \quad i \leqslant j$$

$$= \frac{j\,(N-i)}{N}, \quad i \geqslant j$$

Proof :

The proof is as given in Jain [11]

Since for sufficiently small h, D and D + PM are irreducible and monotone

and $D + PM \geqslant D$

we have

$$(D + PM)^{-1} \leqslant D^{-1}$$

From (18) we obtain

$$(20) \qquad \| E \| \leqslant \| D^{-1} \quad T \|$$

We next obtain a bound on the local truncation error. Since for fixed x_k,

$$\lim_{h \to 0} \frac{B_{1,k}}{h^4} = - \frac{1}{6} x_k^{-2} , \quad k = 2(1)\ N-1$$

and

$$\lim_{h \to 0} \frac{B_{2,k}}{h^4} = \frac{1}{6} x_k^{-1} , \quad k = 2(1)\ N-1$$

It follows that for sufficiently small h,

$$(21) \qquad |B_{1,k}| < \frac{1}{3} h^4 x_k^{-2} , \qquad k = 2(1)\ N-1$$

$$(22) \qquad |B_{2,k}| < \frac{1}{3} h^4 x_k^{-1} , \qquad k = 2(1)\ N-1$$

We assume that

$$|f'| \leqslant N_1$$

$$x\, |f''| \leqslant N_2 , \qquad 0 < x \leqslant 1 .$$

where

N_1 and N_2 are suitable positive constants. With the help of (21), (22) from (9) we obtain for sufficiently small h,

$$|t_k| \leqslant Ch^4 x_k^{-2}$$

where

$$C = \frac{7N_1 + 4N_2}{12}$$

From (20) we obtain

$$|e_i| \leqslant \sum_{j=1}^{N-1} d_{ij}^{-1} \, |t_j|$$

$$\leqslant Ch^{4-2} \left[\sum_{j=1}^{i} d_{ij.}^{-1} \, j^{-2} + \sum_{j=i+1}^{N-1} d_{ij}^{-1} \, j^{-2} \right]$$

$$\leqslant Ch^2 \left[(N-i) \sum_{j=1}^{i} \frac{j.j^{-2}}{N} + i \sum_{j=i+1}^{N-1} \frac{(N-j).\, j^{-2}}{N} \right]$$

$$\leqslant Ch^3 \left[(N-i) \int_1^i \frac{1}{j} \, d_j + i \int_i^N (Nj^{-2} - j^{-1}) \, d_j \right]$$

$$\leqslant Ch^3 \left[(N-i) \log i + i \left\{ N \frac{(N^{-1} - i^{-1})}{(-1)} - \log \frac{N}{i} \right\} \right]$$

$$Ch^3 \left[N \log i - i \log N - i + N \right], \quad i = 1(1)\, N-1$$

Let us consider the following $f(x)$ as a continuous function of $x \in [1, N-1]$

$$f(x) = N \log x - x \log N - x + N$$

For max f(x)

$$f'(x) = \frac{N}{i} - \log N-1 = 0$$

gives

$$i = x = \frac{N}{1 + \log N}$$

$$\| E \| = \max_{1 \leqslant i \leqslant N-1} |e_i| = Ch^2 \left[\log \left(\frac{N}{1 + \log N} \right) \right]$$

$$= O\,(h \log h)^2$$

5. Numerical Illustration

To illustrate our method and its $O\,(h \log h)^2$ convergence, we consider the following example.

Example :

We consider the linear differential equation

$$xy'' = (2\beta -1)\, x^{\beta -1} + \beta(\beta- 1)\, x^{\beta -1} \log x$$

subjected to boundary conditions

$$y(0) = 0, \quad y(1) = 0$$

exact solution is $y = x^{\beta} \log x$

For $N = 2^k$, $k = 2(1)6$, the corresponding value of $||E||$ are shown in table.

TABLE

N	$\|\|E\|\|$
$\beta = 3.00$	
4	2.3 (-2)
8	7.4 (-3)
16	2.0 (-3)
32	5.0 (-4)
64	1.2 (-4)
$\beta = 4.00$	
4	5.3 (-2)
8	2.2 (-2)
16	8.8 (-3)
32	3.2 (-3)
64	1.1 (-3)

CHAPTER - III

A SECOND-ORDER FINITE DIFFERENCE METHOD FOR A CLASS OF SINGULAR TWO POINT BOUNDARY VALUE PROBLEMS

1. Introduction

We generalize the differential equation of Chapter I (equation (1)) to

(1) $$\frac{d}{dx}\left(p(x) \frac{dy(x)}{dx} \right) = f(x,y), \quad 0 < x \leqslant 1$$

with boundary conditions

$$y(0) = A$$

$$y(1) = B$$

where we assume that the function $p(x)$ satisfies

(2) (i) $p(x) > 0$ in $(0,1)$

(ii) $p \in C'(0,1)$ and

(iii) $\frac{1}{p} \in L' [0,1]$

We also consider the conditions

$p'(x)$, $p'''(x) > 0$

and $p''(x) < 0$ (see [3,4])

It is easy to very that the particular choice $p(x) = x^{\alpha}$, $0 \leqslant \alpha < 1$, does in fact satisfy all the conditions (2).

Our object in the present chapter is to discuss the construction of three point finite difference approximation and its convergence under appropriate conditions for the class of singular non-linear two point boundary value problem (1). In Section 2, we discuss the construction of our finite difference method and proved its second order convergence in Section 4.

2. The Finite Difference Method

For a + ve integer $N \geqslant 2$ consider uniform mesh over closed interval $[0,1]$: $0 = x_0 < x_1 < x_2 \ldots \ldots < x_N = 1$ with $x_k = kh$. Let $y_k = y(x_k)$, $f_k = f(x_k, y_k)$ etc.

we set

$$z(x) = p(x)y' ;$$

Integrating (1) from x_k to x, we obtain

(3) $$z(x) = z_x + \int_{x_k}^{x} f(t)\, dt$$

where

$$f(t) = f(t, y(t))$$

Dividing (3) by p(x) and integrating from x_k to x_{k+1} and interchanging the order of integration, we obtain

(4) $$y_{k+1} - y_k = J_k . Z_k + \int_{x_k}^{x_{k+1}} \left(\int_{t}^{x_{k+1}} \frac{1}{p(x)} dx \right) f(t)\, dt$$

where we have set

(5) $$J_k = \int_{x_k}^{x_{k+1}} \frac{1}{p(x)}\, dx$$

Let

(6) $$P(x) = \int_{0}^{x} \frac{1}{p(t)} dt \qquad \forall\, x \in [0,1]$$

So (5) becomes

(7) $$J_k = P(x_{k+1}) - P(x_k)$$

In an analogus manner, we obtain

$$(8) \qquad y_k - y_{k-1} = Z_k . J_{k-1} - \int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} \frac{1}{p(x)} dx \right) f(t)\, dt.$$

Eliminating Z_k from (4) and (8) we obtain the identity:

$$(9) \qquad \frac{y_{k+1} - y_k}{J_k} - \frac{y_k - y_{k-1}}{J_{k-1}} = \frac{I_k^{+}}{J_k} + \frac{I_k^{-}}{J_{k-1}} , \quad k = 1(1)\, N-1$$

where we have set

$$I_k^{\pm} = \int_{x_k}^{x_{k \pm 1}} \left(P(x_{k \pm 1}) - P(t) \right) f(t)\, dt$$

We assume that $P(x) \in L' [0,1]$

Identity (9) is our basic result from which methods of various orders can be obtained for the two point boundary value problem (1). However, we shall be interested here in obtaining a method of order two which will be based on 1 evaluation of f.

By Taylor's expansion of f about x_k, from (9) we obtain

$$(10) \quad -\frac{1}{J_{k-1}} y_{k-1} + \left(\frac{1}{J_{k-1}} + \frac{1}{J_k} \right) y_k - \frac{1}{J_k} y_{k+1}$$

$$+ B_{o,k} f_k + t_k = 0,$$

$$k = 1(1)\ N-1$$

where

$$(11) \quad t_k = B_{1,k} f'_k + \frac{1}{2} B_{2,k} f''(\xi_k)$$

$$\xi_k \in (x_{k-1}, x_{k+1})$$

and

$$(12) \quad B_{m,k} = \frac{A^{+}_{m,k}}{J_k} + \frac{A^{-}_{m,k}}{J_{k-1}}, \quad m = 0,1,2$$

$$A^{\pm}_{0,k} = \int_{x_k}^{x_{k\pm 1}} (P(x_{k \pm 1}) - P(t))\, dt$$

(13) $$A^{\pm}_{1,k} = \int_{x_k}^{x_{k\pm 1}} (P (x_{k\pm 1}) - P(t))(t-x_k)dt$$

$$A^{\pm}_{2,k} = \int_{x_k}^{x_{k\pm 1}} (P (x_{k \pm 1}) - P(t)) (t-x_k)^2 dt$$

A finite difference method can now be based on the discretization (10) of differential equation involving one evaluation of f. In section 4 we show that, under suitable conditions, our method based on (10) is $O(h^2)$ convergent.

3. Matrix Formulation of Our Finite Difference Method

It is convenient to describe the above method in matrix form. Let $D = (d_{i,j})_{i,j=1}^{N-1}$ denote the tridiagonal matrix with

$$d_{k,k-1} = -\frac{1}{J_{k-1}},$$

$$k = 2(1)\ N-1$$

(14) $$d_{k,k} = \frac{1}{J_{k-1}} + \frac{1}{J_k},$$

$$k = 1(1)\ N-1$$

$$d_{k,k+1} = -\frac{1}{J_k},$$

$$k = 1(1)\ N-2$$

let

$P = (p_{ij})_{i,j=1}^{N-1}$ denote the diagonal matrix with

$$p_{k,k} = B_{o,k}, \qquad k = 1(1)\ N-1$$

let

$$Q = (q_1,\ 0,\ \ldots\ \ldots\ \ldots\ \ldots\ 0,\ q_{N-1})^T$$

where

$$q_1 = \frac{A}{J_0}$$

$$q_{N-1} = \frac{B}{J_{N-1}}$$

Also let

$$F(Y) = (f_1, \ldots \ldots \ldots f_{N-1})^T$$

$$Y = (y_1, \ldots \ldots \ldots y_{N-1})^T$$

and $$T = (t_1, \ldots \ldots t_{N-1})^T$$

Thus the discretization (10) can be expressed in matrix form:

(15) $DY + PF(Y) + T = Q$

The method now consists of finding an approximation $\widetilde{Y}$ for Y by solving the $(N-1) \times (N-1)$ system :

(16) $D\widetilde{Y} + PF(\widetilde{Y}) = Q$

We note that our coefficient matrix D is symmetric. In case the differential equation is linear (16) is tridiagonal linear system; in case of non-linear differential equation, the non-linear system can be solved by Newton - Raphson method and an adaption of Gauss - elimination for tri-diagonal linear systems.

4. Convergence of the Method

We next discuss the convergence of our method showing that, under suitable conditions, our method is $O(h^2)$ convergent.

let

$$E = (e_1, \ldots \ldots e_{N-1})^T$$

$$= \widetilde{Y} - Y$$

we may write

(17) $$f(x_k, \widetilde{y}_k) - f(x_k, y_k) = e_k U_k ,$$

$$k = 1(1) N-1$$

for suitable U_k, now

(18) $F(\tilde{Y}) - F(Y) = ME$

where

$M = (m_{i,j})_{i,j=1}^{N-1}$ is diagonal matrix with

(19) $m_{k,k} = U_k \quad , \quad k = 1(1)\ N-1$

we also note that $U_k \geqslant 0$

With the help of (18), from (15) and (16) we obtain the error equation :

(20) $(D+PM)E = T$

To show that our method is $O(h^2)$ convergent we first establish the following lemmas.

<u>Lemma</u> 1 :

$B_{o,k} > 0 \quad$ for $\quad k = 1(1)\ N-1$

<u>Proof</u> :

$$B_{o,k} = \frac{\overset{+}{A}_{o,k}}{J_k} + \frac{\overset{-}{A}_{o,k}}{J_{k-1}}$$

$$(21) \qquad B_{o,k} = \frac{P(x_{k+1}) \int_{x_k}^{x_{k+1}} dt - \int_{x_k}^{x_{k+1}} P(t)\, dt}{P(x_{k+1}) - P(x_k)} + \frac{\int_{x_{k-1}}^{x_k} P(t)\, dt - P(x_{k-1}) \int_{x_{k-1}}^{x_k} dt}{P(x_k) - P(x_{k-1})}$$

Making the transformation $t = x_k + hU$ and expanding the expression of R.H.S. of (21) by Taylor's expansion about the point x_k, we obtain for sufficiently small h,

$$B_{o,k} \sim h ,$$

Hence,

$$B_{o,k} > 0 \text{ for sufficient small } h.$$

This completes the proof the lemma.

<u>Lemma</u> 2 :

For sufficiently small h,

$$(22) \qquad |B_{1,k}| < \frac{1}{6} h^3 \frac{p_k'}{p_k} , \ k = 1(1) N-1$$

(23) $\quad \left| B_{2,k} \right| < \frac{1}{6} h^3 \, , \quad k = 1(1)\,N-1$

<u>Proof</u> :

For fixed $x_k = kh$

$$B_{1,k} = \frac{A^{+}_{1,k}}{J_k} + \frac{A^{-}_{1,k}}{J_{k-1}}$$

(24)
$$B_{1,k} = \frac{\int_{x_k}^{x_{k+1}} \left(P(x_{k+1}) - P(t)\right)(t-x_k)\,dt}{P(x_{k+1}) - P(x_k)} + \frac{\int_{x_{k-1}}^{x_k} \left(P(t) - P(x_{k-1})\right)(t-x_k)\,dt}{P(x_k) - P(x_{k-1})}$$

Making the transformation $t = x_k + hU$ and expanding the expression on R.H.S. of (24) by Taylor's expansion about the point x_k, we obtain

(25) $$\lim_{h\to 0} \frac{B_{1,k}}{h^3} = - \frac{1}{12} \frac{p_k'}{p_k} ,$$

$$k = 1(1)\ N-1$$

It follows that for sufficiently small h,

$$|B_{1,k}| < \frac{1}{6} h^3 \frac{p_k'}{p_k}$$

Similarly it can be shown

(26) $$\lim_{h\to 0} \frac{B_{2,k}}{h^3} = \frac{1}{12} , \quad k = 1(1)\ N-1$$

It follows that for sufficiently small h,

$$|B_{2,k}| < \frac{1}{6} h^3$$

Lemma 3 :

The inverse of the matrix D is given by

(27) $$d_{ij}^{-1} = \frac{P(x_i) \ [\ P(x_N) - P(x_j)\ ]}{P(x_N)} , \quad i \leqslant j$$

$$= \frac{P(x_j) \ [\ P(x_N) - P(x_i)\ ]}{P(x_N)} , \quad i \geqslant j$$

<u>Proof</u> :

(See Appendix [1])

We assume that

$$|f'| \leqslant N_1 ,$$

$$\frac{p(x)}{p'(x)} \cdot |f''| \leqslant N_2, \quad 0 < x \leqslant 1 ,$$

where N_1 and N_2 are suitable positive constants.

With the help of (25) and (26) from (11) we obtain for sufficiently small h,

$$(28) \qquad |t_k| \leqslant Ch^3 \frac{p'(x_k)}{p(x_k)}$$

where

$$C = \frac{2N_1 + N_2}{12}$$

Now since D and D + PM are irreducible and monotone for sufficiently small h, and since D + PM $\geqslant$ D

it follows that

$$(D + PM)^{-1} \leqslant D^{-1}$$

From (20) we obtain

$$(29) \qquad ||E|| \leqslant ||D^{-1} \quad T||$$

With the help of (28) and (27), from (29) we obtain for i = 1(1) N-1

$$(30) \qquad |e_i| \leqslant \sum_{j=1}^{N-1} d_{i,j}^{-1} \; |t_j| \, .$$

$$(31) \qquad \leqslant Ch^3 \Big[\frac{P(x_N) - P(x_i)}{P(x_N)} \sum_{j=1}^{i} P(x_j) \frac{p'(x_j)}{p(x_j)}$$

$$+ \frac{P(x_i)}{P(x_N)} \sum_{j=i+1}^{N-1} \left(P(x_N) - P(x_j) \right) \frac{p'(x_j)}{p(x_j)}$$

It is easy to establish the inequality :

$$(32) \qquad h \sum_{j=1}^{i} \frac{P(x_j)\; p'(x_j)}{p(x_j)} < \int_0^{x_i} \frac{P(x)\; p'(x)}{p(x)} \, dx$$

$$< P(x_i) \log p(x_i) - \int_0^{x_i} \frac{\log p(x)}{p(x)} \, dx$$

Again

$$(33) \qquad h \sum_{J=i+1}^{N-1} \left(P(x_N) - P(x_j) \right) \frac{p'(x_j)}{p(x_j)}$$

$$< \int_{x_i}^{x_N} (P(x_N) - P(x)) \; \frac{p'(x)}{p(x)} \, dx$$

$$< (P(x_i) - P(x_N)) \log p(x_i) +$$

$$\int_{x_i}^{x_N} \frac{\log p(x)}{p(x)} dx .$$

With the help of (32) and (33) from (31) we obtain

(34) $$|e_i| \leqslant \frac{Ch^2}{P(x_N)} \left[(P(x_i) - P(x_N)) \int_0^{x_i} \frac{\log p(x)}{p(x)} dx + \right.$$

$$\left. P(x_i) \int_{x_i}^{x_N} \frac{\log p(x)}{p(x)} dx \right) .$$

Lemma 4 :

(35) $\log p(x) \leqslant p(x)$

for $p(x) > 0$

and $x \in (0,1)$

Proof :

<u>Case</u> I

when $0 < p(x) < 1$

(35) can be written as

$$p(x) \leqslant e^{p(x)}$$

$$p(x) \leqslant 1 + p(x) + \frac{[p(x)]^2}{2!} + \ldots.$$

which is true

<u>Case</u> II when $p(x) > 1$

then

$$p(x) < e^{p(x)}$$

which is true

This completes the proof of lemma 4 :

So we can write (34) as

$$(36) \qquad |e_i| \leqslant \frac{Ch^2}{P(x_N)} [(P(x_i) - P(x_N))x_i + P(x_i)$$

$$(x_N - x_i)]$$

$$(37) \qquad \leqslant \frac{Ch^2}{P(x_N)} [P(x_i) - x_i P(x_N)]$$

let

$$f(i) = P(ih) - ih \cdot P_N'$$

for Maximum

$$f'(i) = h P' (ih) - hP_N = 0$$

$$\frac{1}{p(ih)} = P_N$$

$$ih = p^{-1}\left(\frac{1}{P_N}\right)$$

$$i = \frac{1}{h}\, p^{-1}\left(\frac{1}{P_N}\right)$$

$$f''(i) = h^2 \quad P''(ih)$$

since $P(x) = \int_0^x \frac{1}{p(x)}\,dx$

$$P'(x) = \frac{1}{p(x)}$$

$$P''(x) = -\frac{1}{p^2(x)}\, p'(x) < 0$$

So $f''(i) < 0$.

so $i = \frac{1}{h}\, p^{-1}\left(\frac{1}{P_N}\right)$ gives max. error bound

So (37) becomes

$$\|E\| = \max_{1 \leqslant i \leqslant N-1} |e_i|$$

$$\|E\| \leqslant \frac{Ch^2}{P(x_N)} \left[P\left(p^{-1}\left(\frac{1}{P_N}\right)\right) - P_N\, p^{-1}\left(\frac{1}{P_N}\right) \right]$$

$$\|E\| = O(h^2)$$

APPENDIX [1]

The inverese of the matrix D, where

$D = (d_{i,j})_{i,j=1}^{N-1}$ denote the tridiagonal matrix with

$$d_{k,k-1} = -\frac{1}{J_{k-1}} \quad , \quad k = 2(1)\,N-1$$

(1) $$d_{k,k} = \frac{1}{J_{k-1}} + \frac{1}{J_k} \quad , \quad k = 1(1)\,N-1$$

$$d_{k,k+1} = -\frac{1}{J_k} \quad , \quad k = 1(1)\,N-2$$

(2) With

$$J_k = \int_{x_k}^{x_{k+1}} \frac{1}{p(x)}\,dx$$

$$= P(x_{k+1}) - P(x_k)$$

where

(3) $$P(x) = \int_{o}^{x} \frac{1}{p(x)}\,dx$$

let $D^{-1} \doteq (d_{i,j}^{-1})_{i,j=1}^{N-1}$

On multiplying the i^{th} row of D with the j^{th} coloumn of D^{-1} we obtain the following difference equations, for k = 1(1) N-1

(4) $$-\frac{d^{-1}_{i-1,j}}{J_{i-1}} + d^{-1}_{i,j}\left(\frac{1}{J_{i-1}} + \frac{1}{J_i}\right) - \frac{d^{-1}_{i+1,j}}{J_i} = 0 \ , \quad i = 2(1)\ j-1$$

(5) $$d^{-1}_{i,j}\left(\frac{1}{J_o} + \frac{1}{J_1}\right) - \frac{d^{-1}_{2,j}}{J_1} = 0, \quad j \neq 1$$

(6) $$-\frac{d^{-1}_{j-1,j}}{J_{j-1}} + d_{j,j}\left(\frac{1}{J_{j-1}} + \frac{1}{J_j}\right) - \frac{d^{-1}_{j+1,j}}{J_j} = 1$$

(7) $$-\frac{d^{-1}_{i-1,j}}{J_{i-1}} + d^{-1}_{i,j}\left(\frac{1}{J_{i-1}} + \frac{1}{J_i}\right) - \frac{d^{-1}_{i+1,j}}{J_i} = 0 \ , \quad i = j+1(1)\ N-1$$

$$(8) \qquad - \frac{d^{-1}_{N-2,j}}{J_{N-2}} + d^{-1}_{N-1,j} \left(\frac{1}{J_{N-2}} + \frac{1}{J_{N-1}} \right) = 0 \quad ,$$

$j \neq N$

solving (4) subject to (5) we obtain

$$(9) \qquad d^{-1}_{i,j} = B(j) \; P(x_i) \quad , \qquad i \leqslant j$$

where B(j) is a parameter depending on j.

Again solving (7) subject to (8) we obtain

$$(10) \qquad d^{-1}_{i,j} = D(j) \left[P(x_i) - P(x_N) \right] , \; i \geqslant j$$

where D(j) is a parameter depending on j.

Now in order that $d^{-1}_{j,j}$ is identical as given by (9) and (10) , we must have

$$(11) \qquad B(j) \; P(x_j) = D(j) \left[P(x_j) - P(x_N) \right]$$

Also in order that $d^{-1}_{i,j}$ as given by (9) and (10) satisfies (6), we obtain

$$(12) \quad - \frac{B(j)\ P(x_{j-1})}{J_{j-1}} + B(j)\ P(x_j)$$

$$\left(\frac{1}{J_{j-1}} + \frac{1}{J_j} \right)$$

$$- \frac{D(j) \left[P(x_{j+1}) - P(x_N) \right]}{J_j} = 1$$

Solving (11) and (12) we obtain

$$(13) \quad B(j) = \frac{\left[P(x_N) - P(x_j) \right]}{P(x_N)}$$

$$(14) \quad D(j) = - \frac{P(x_j)}{P(x_N)}$$

Substituting for B(j) and D(j) from (13) and (14) in (9) and (10), we obtain

$$(15) \qquad d^{-1}_{i,j} = \frac{P(x_i) \left[P(x_N) - P(x_j) \right]}{P(x_N)}, \quad i \leqslant j$$

$$= \frac{P(x_j) \left[P(x_N) - P(x_i) \right],}{P(x_N)} \quad i \geqslant j$$

Example :

Let $\quad p(x) = x^{\alpha}$

$$P(x) = \frac{x^{1-\alpha}}{1-\alpha}$$

So Matrix $D^{-1} = (d^{-1}_{i,j})^{N-1}_{i,j=1}$,

together with $x_N = 1$, becomes

$$d^{-1}_{i,j} = x_i^{1-\alpha} (1-x_j^{1-\alpha}) \;/\; (1-\alpha), \quad i \leqslant j$$

$$= x_j^{1-\alpha} (1 - x_i^{1-\alpha}) \;/\; (1-\alpha), \quad i \geqslant j$$

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